Boundary Conformal Field Theory and Fusion Ring Representations

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Abstract

To an RCFT corresponds two combinatorial structures: the 1-loop partition function of a closed string (the amplitude of a torus, sometimes called a *modular invariant*), and the 1-loop partition function of an open string (a representation of the fusion ring called a *NIM-rep* or equivalently a *fusion graph*). In this paper we develop some basic theory of NIM-reps, obtain several new NIM-rep classifications, and compare them with the corresponding modular invariant classifications. Among other things, we make the following fairly disturbing observation: there are infinitely many (WZW) modular invariants which do not correspond to any NIM-rep. The resolution could be that those modular invariants are physically sick. Is classifying modular invariants really the right thing to do? For current algebras, the answer seems to be: Usually but not always. For finite groups à la Dijkgraaf-Vafa-Verlinde-Verlinde, the answer seems to be: Rarely.

1. Introduction

For many reasons, not the least of which is open string theory, we are interested in boundary conformal field theory. Although it has apparently never been established that bulk RCFT necessarily requires for its consistency that there be a compatible system of boundary conditions, the conventional wisdom seems to be that otherwise the RCFT would have sick operator product expansion. In any case, a boundary CFT would seem to be a relatively accessible halfway-point between constructing the full CFT from a chiral CFT(=vertex operator algebra).

Cardy [1] was the first to explain how conformally invariant boundary conditions in CFT are related to fusion coefficients. In particular, given a bulk CFT and a choice of (not necessarily maximal) chiral algebra, consider the set of (conformally invariant) boundary conditions which don't break the chiral symmetry. These should be spanned by the appropriate Ishibashi states $|\mu\rangle\rangle$, labelled by the spin-zero primary fields $\phi(\mu,\overline{\mu})$ in the theory. We know [2] these boundary states need not be linearly independent, but we should be able to find a (unique) $\mathbb{Z}_{\geq}$-basis $|x\rangle \in \mathcal{B}$ for them, equal in number to the Ishibashi states. Then the 1-loop vacuum amplitude $\mathcal{Z}_{x|y}$, where the two edges of the cylinder are decorated with boundary conditions $|x\rangle, |y\rangle \in \mathcal{B}$, can be expanded in terms of the chiral characters χ_λ:

$$\mathcal{Z}_{x|y} = \sum_\lambda \mathcal{N}_{\lambda x}^y \, \chi_\lambda$$

Cardy explained that these coefficients $\mathcal{N}_{\lambda x}^y$ define a representation of the chiral fusion ring with nonnegative integer matrices. Strictly speaking, Cardy only considered the diagonal theory given by the modular invariant partition function $\mathcal{Z} = \sum_\mu |\chi_\mu|^2$. The more general theory has been developed by e.g. Pradisi et al (see e.g. [3,4] for a good review), Fuchs–Schweigert (see e.g. [5]), and Behrend et al (see e.g. [6]). We will review and axiomatise the combinatorial essence of this theory below in Section 3, under the name *NIM-reps*.

In a remarkable paper, Di Francesco–Zuber [7] sought a generalisation of the A-D-E pattern of $\widehat{\mathrm{sl}}(2)$ modular invariants, by assigning graphs to RCFT. They were (largely empirically) led to introduce what we now will call *fusion graphs*. Over the years the definition was refined, and their relations to the lattice models of statistical mechanics, structure constants in CFT, etc were clarified (see the enchanting review in [8]). In particular, their connection with NIM-reps is now fully understood (see e.g. [6]).

The torus partition function (=modular invariant) and the cylinder partition function (=NIM-rep) of an RCFT should be compatible. Roughly, the eigenvalues of the NIM-rep matrices $(\mathcal{N}_\lambda)_{xy} = \mathcal{N}_{\lambda x}^y$ should be labeled by the Ishibashi states, and the Ishibashi states should be labeled by the spin-0 primaries, i.e. the diagonal ($\lambda = \mu$) terms in the modular invariant $\mathcal{Z} = \sum_{\lambda,\mu} M_{\lambda\mu} \, \chi_\lambda \, \chi_\mu^*$. (Strictly speaking, all this assumes a choice of 'pairing' or 'gluing automorphism' ω — see §2.2 below.)

We call a modular invariant *NIMmed* if it has a compatible NIM-rep; otherwise we call it *NIM-less*. In this way, we can use NIM-reps to probe lists of modular invariants. After all, the definition (see §2.2) of modular invariants isolates only certain features of RCFT, and it is not at all obvious that classifying them is really the right thing to do.

NIM-reps and modular invariants, and their compatibility condition, also appear very naturally in the context of braided subfactor theory (see e.g. [9,10] for reviews of this remarkable picture, due originally to Ocneanu). The term 'NIM-rep' [10] is short for 'nonnegative integer matrix representation'.

Even if we restrict attention to the current(=affine Kac-Moody) algebras, i.e. WZW theories, very little is known about NIM-rep classifications. The $\widehat{\mathrm{sl}}(2)$ theories at all levels k, and all $\widehat{\mathrm{sl}}(n)$ at level 1, are all that have been done [7,6], although Ocneanu [11] has announced a classification of the subset of $\widehat{\mathrm{sl}}(3)$ and $\widehat{\mathrm{sl}}(4)$ NIM-reps (any level) of relevance to subfactors. Although there isn't a perfect match with the corresponding modular invariant classifications, the relation between what superficially seem to be distinct mathematical problems is remarkable.

Our two main results are:

- We classify the NIM-reps for all $\widehat{\mathrm{sl}}(n)$ and $\widehat{\mathrm{so}}(n)$ at level 2. Those of $\widehat{\mathrm{sl}}(n)$ match up well with the corresponding modular invariant classification; those of $\widehat{\mathrm{so}}(n)$ dramatically do not, and in fact most $\widehat{\mathrm{so}}(n)$ level 2 modular invariants are NIM-less.
- We develop the basic theory of NIM-reps (see especially Theorem 3 below), and find striking similarities with modular invariants (compare Theorem 1).

In §3.4 we discuss the rationality and nonnegativity of the coefficients $\mathcal{M}_{\lambda\mu}^{\nu}$ of the Pasquier algebra and of the dual Pasquier algebra $\widehat{\mathcal{N}}_{xy}^{z}$. In §4 we give the level 1 NIM-rep classifications for all current algebras. We relegate the (unpleasant) proofs of the level 2 classifications and Theorem 3 to the Appendix.

The reader less interested in the details may wish to jump to §6, where we find two simple *NIM-less* modular invariants, then to §7 where we explain using the example of $\widehat{\mathrm{sl}}(3)$ level 8 how the results of §3.3 come together to yield NIM-rep classifications, and finally move to the conclusion, §8, where we give some concluding thoughts and speculations.

What do all these NIM-less modular invariants tell us? Either this picture of the relation of boundary and bulk CFT is too naive (e.g. perhaps the change-of-coordinate matrix U in (3.1) isn't unitary), or there are infinitely many modular invariants which aren't realised as the partition function of a CFT.

What about NIM-reps for higher-rank algebras and levels? We get good control over the eigenvalues of the fusion graphs. Much more difficult is, given these eigenvalues, to draw the possible fusion graphs. We know (proved below) that there will only be finitely many of these, but based on considerations given in **(6)** in the concluding section, we expect that number to be typically quite large. The classes presently worked out are atypical, because the critical Perron-Frobenius eigenvalues involved are ≤ 2. As the eigenvalues rise, we expect the number of NIM-reps to grow out of control. In other words, we expect that classifying NIM-reps is probably hopeless (and pointless) for all but the smallest ranks and levels.

On the other hand, [12] suggests that the modular invariant situation for $\widehat{\mathrm{so}}(n)$ level 2 is quite atypical, and that we can expect that all modular invariants for most current algebras $X_{r,k}$ are related to Dynkin diagram symmetries. The corresponding NIM-reps would then be fairly well understood (see e.g. [13,14] and references therein); in particular they are probably all NIMmed. The situation however will probably be much worse for the modular invariants coming from other (i.e. non-WZW) chiral algebras, e.g. the untwisted sector in holomorphic orbifolds by finite groups [15] — see e.g. §6.

There is a tendancy in the literature to focus only on $\widehat{\mathrm{sl}}(n)$ (although [16] briefly discussed NIM-reps for $\widehat{G}_2$ level k). This perhaps is a mistake — $\widehat{\mathrm{sl}}(n)$ is very special, and this limited perspective leads to incorrect intuitions as to characteristic WZW or RCFT behaviour. We see that here: for instance the NIM-rep vrs modular invariant situation for $\widehat{\mathrm{so}}(n)$ level 2 is quite remarkable, and considerably more interesting than that for $\widehat{\mathrm{sl}}(n)$ level 2.

2. Review: Fusion rings and modular invariants

2.1. Modular data of the RCFT.

The material of this subsection is discussed in more detail in the reviews [17,18].

As is well-known, the RCFT characters $\chi_\lambda(q)$ yield a finite-dimensional unitary representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$, given by the natural action of $\mathrm{SL}_2(\mathbb{Z})$ on $\tau = \frac{1}{2\pi \mathrm{i}}\ln q$. Denote by S and T the matrices associated to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then T is diagonal, and S is symmetric. The rows and columns of S and T are parametrised by the primary fields $\lambda \in P_+$. One of these, the 'vacuum' 0, is distinguished.

In this paper we will be primarily interested in the data coming from current algebras $\widehat{\mathfrak{g}}$ ($\mathfrak{g}$ simple), i.e. associated to Wess-Zumino-Witten models. However, unless otherwise stated, everything here holds for arbitrary RCFT.

We will assume for convenience that $S_{\lambda 0} > 0$ — this holds in particular for unitary RCFTs, such as the WZW models. The changes required for nonunitary RCFT consist mainly of replacing some appearances of the vacuum with the unique primary $o \in P_+$ with minimal conformal weight. Then $S_{\lambda o} > 0$. In a unitary theory, we have $o = 0$. The ratio $S_{\lambda o}/S_{0o}$ is called the *quantum-dimension* of λ, and plays a central role.

The matrix S^2 is a permutation matrix C, called *charge-conjuation*. It obeys $C0 = 0$, $T_{C\lambda,C\lambda} = T_{\lambda\lambda}$, and corresponds to complex conjugation:

$$S_{C\lambda,\mu} = S_{\lambda,C\mu} = S^*_{\lambda\mu} \tag{2.1}$$

The fusion coefficients $N_{\lambda\mu}^{\nu}$ of the theory are given by Verlinde's formula [19]:

$$N_{\lambda\mu}^{\nu} = \sum_{\gamma\in P_+} \frac{S_{\lambda\gamma}\,S_{\mu\gamma}\,S^*_{\nu\gamma}}{S_{0\gamma}} \in \mathbb{Z}_{\geq} := \{0,1,2,\ldots\} \tag{2.2}$$

Let N_λ denote the *fusion matrix*, i.e. the matrix with entries $(N_\lambda)_{\mu\nu} = N_{\lambda\mu}^{\nu}$. Then $N_0 = I$, $N_{C\lambda} = N_\lambda^t$, and

$$N_\lambda\, N_\mu = \sum_{\nu\in P_+} N_{\lambda\mu}^{\nu} N_\nu \tag{2.3}$$

We use the term *modular data* for any matrices S and T obeying these conditions. The ring with preferred basis P_+ and structure constants $N_{\lambda\mu}^{\nu}$ is called the *fusion ring*. For example, modular data and a fusion ring exist for every choice of current algebra $\widehat{\mathfrak{g}} = X_r^{(1)}$ and positive integer k (called the *level*) — of course this is precisely what arises in the WZW models. At times we will abbreviate this to $X_{r,k}$. The primaries $\lambda \in P_+$ for this WZW modular data consist of the level k integrable highest weights $\lambda = \lambda_1\Lambda_1 + \cdots + \lambda_r\Lambda_r$, where the basis vectors Λ_i are called fundamental weights. See e.g. Ch.13 of [20] for more details. Explicit formulas for $S_{\lambda\mu}$ are given in [20]; see also [21].

The quantum-dimensions in (unitary) RCFT obey $S_{\lambda 0}/S_{00} \geq 1$. When it equals 1, λ is called a *simple-current* [22]. Then N_λ will be a permutation matrix, corresponding to a permutation J of P_+, and there will be a phase $Q_J : P_+ \to \mathbb{Q}$ such that

$$S_{J\mu,\nu} = e^{2\pi \mathrm{i}\, Q_J(\nu)}\, S_{\mu\nu} \tag{2.4}$$

The simple-currents form an abelian group, under composition of permutations. Note that

$$N_{J\mu,J'\nu}^{JJ'\gamma} = N_{\mu\nu}^{\gamma} \tag{2.5a}$$
$$N_{C\mu,C\nu}^{C\gamma} = N_{\mu\nu}^{\gamma} \tag{2.5b}$$

for any simple-currents J, J', where C as usual is charge-conjugation.

For example, for $A_{1,k}$ we may take $P_+ = \{0, 1, \ldots, k\}$ (the value of the Dynkin label λ_1), and then the S matrix is $S_{ab} = \sqrt{\frac{2}{k+2}}\, \sin(\pi\, \frac{(a+1)\,(b+1)}{k+2})$. Charge-conjugation C is trivial here, but $j = k$ is a simple-current corresponding to permutation $Ja = k - a$ and phase $Q_j(a) = a/2$. The fusion coefficients are given by

$$N_{ab}^c = \begin{cases} 1 & \text{if } c \equiv a + b \pmod 2 \text{ and } |a - b| \leq c \leq \min\{a + b, 2k - a - b\} \\ 0 & \text{otherwise} \end{cases}$$

Write ξ_N for the root of unity $\exp[2\pi \mathrm{i}/N]$. A fundamental symmetry of modular data is a certain generalisation of charge-conjugation. For any RCFT, the entries $S_{\lambda\mu}$ are sums of roots of unity ξ_N^m, all divided by a common integer L. For example for $\mathrm{sl}(n)_k$ we can take $N = 4n\,(n + k + 1)$. We say that the entries $S_{\lambda\mu}$ lie in the *cyclotomic number field* $\mathbb{Q}[\xi_N]$. The automorphisms $\sigma \in \mathrm{Gal}(\mathbb{Q}[\xi_N]/\mathbb{Q})$ of this field preserve by definition both multiplication and addition and fix the rational numbers; they are parametrised by an integer ℓ coprime to N (more precisely, the action of σ_ℓ is uniquely determined by the relation $\sigma_\ell(\xi_N^m) = \xi_N^{m\ell}$, so really ℓ is defined modulo N). To each such integer ℓ, i.e. each automorphism σ_ℓ, there is a permutation σ_ℓ of P_+ and a choice of signs $\epsilon_\ell(\lambda) = \pm 1$, such that [23]

$$\sigma_\ell(S_{\lambda\mu}) = \epsilon_\ell(\lambda)\, S_{\sigma_\ell(\lambda),\mu} = \epsilon_\ell(\mu)\, S_{\lambda,\sigma_\ell(\mu)} \tag{2.6}$$

This Galois symmetry may sound complicated, but that could be due only to its unfamiliarity. It plays a central role in the theory of modular data, modular invariants, and NIM-reps (see e.g. §7 below), and makes accessible problems which have no right to be so. Algorithms for this Galois symmetry, for the current algebras, are explicitly worked out in [21].

An important ingredient of the theory is that of *fusion-generators*. We call $\Gamma = \{\gamma^{(1)}, \ldots, \gamma^{(g)}\} \subset P_+$ a fusion-generator if to any $\lambda \in P_+$ there is a g-variable polynomial $P_\lambda(x_1, \ldots, x_g)$ such that the fusion matrices obey

$$N_\lambda = P_\lambda(N_{\gamma^{(1)}}, \ldots, N_{\gamma^{(g)}})$$

or equivalently, for any $\lambda, \mu \in P_+$,

$$\frac{S_{\lambda\mu}}{S_{0\mu}} = P_\lambda(\frac{S_{\gamma^{(1)}\mu}}{S_{0\mu}}, \ldots, \frac{S_{\gamma^{(g)}\mu}}{S_{0\mu}}) \tag{2.7}$$

This says that $\gamma^{(1)}, \ldots, \gamma^{(g)}$ generate the fusion ring, and also (we will see) the NIM-reps.

One of the reasons fusion rings for the current algebras are relatively tractible is the existence of small fusion-generators. In particular, because we know that any Lie character for X_r can be written as a polynomial in the fundamental weights $\mathrm{ch}_{\Lambda_1}, \ldots, \mathrm{ch}_{\Lambda_r}$, it is easy to show [24] that $\Gamma = \{\Lambda_1, \ldots, \Lambda_r\} \cap P_+$ is a fusion-generator for any $X_r^{(1)}$ level k. Smaller fusion-generators usually exist however. The question for $\mathrm{sl}(n)_k$ has been studied quite thoroughly in [25]. For example, $\{\Lambda_1\}$ is a fusion-generator for $\mathrm{sl}(n)_k$ iff both

(i) each prime divisor p of $k + n$ satisfies $2p > \min\{n, k\}$, and
(ii) either n divides k, or $\gcd(n, k) = 1$.

More generally, for $\mathrm{sl}(n)_k$ the following are always fusion-generators:

$$\Gamma_{\div} = \{\Lambda_d \,|\, 2d \leq n \text{ and } d \text{ divides } k+n\}$$
$$\Gamma^{\tau}_{\div} = \begin{cases} \{\Lambda_d \,|\, 2d \leq k \text{ and } d \text{ divides } k+n\} & \text{when } k \text{ doesn't divide } n \\ \{k\Lambda_1, \Lambda_d \,|\, 2d \leq k \text{ and } d \text{ divides } k+n\} & \text{when } k \text{ divides } n \end{cases}$$

(Of course, the weight $k\Lambda_1$ in $\Gamma^{\tau}_{\div}$ is a simple-current.) Examples are:

- Λ_1 is a fusion-generator for $\mathrm{sl}(2)_k$ and $\mathrm{sl}(3)_k$, for any level k;
- for $\mathrm{sl}(4)_k$, Λ_1 is a fusion-generator when k is odd, while both $\{\Lambda_1, \Lambda_2\}$ are needed when k is even;
- Λ_1 is a fusion-generator for $\mathrm{sl}(n)_1$ for any n; it's also a fusion-generator for $\mathrm{sl}(n)_2$ when n is odd, while both $\{\Lambda_1, 2\Lambda_1\}$ are needed when n is even.

2.2. Modular invariants and their exponents.

The one-loop vacuum-to-vacuum amplitude of a rational conformal field theory is the modular invariant partition function

$$\mathcal{Z}(q) = \sum_{\lambda,\mu \in P_+} M_{\lambda\mu}\, \chi_\lambda(q)\, \chi_\mu(q)^* \tag{2.8}$$

Definition 1. *By a* modular invariant M *we mean a matrix with nonnegative integer entries and* $M_{00} = 1$, *obeying* $MS = SM$ *and* $MT = TM$.

Two examples of modular invariants are $M = I$ and $M = C$ (of course these may be equal). It is known that for any choice of modular data, the number of modular invariants will be finite [26,10]. We identify the function $\mathcal{Z}$ in (2.8) with its coefficient matrix M.

The coefficient matrix M of an RCFT partition function is a modular invariant, but the converse need not be true. Also, different RCFTs can conceivably have the same modular invariant. *Is the classification of modular invariants the right thing to do?* Is there actually a resemblance between the list of modular invariants, and the corresponding list of RCFTs? Or are we losing too much information and structure by classifying not the full RCFTs, but rather the much simpler modular invariants? We return to these questions in the concluding section, §8.

We have a good understanding now of the modular invariant lists for the current algebras, at least for small rank and/or level. See [12,18] and references therein for the main results and appropriate literature.

The most famous modular invariant list is that of $\widehat{\mathrm{sl}}(2)$, due to Cappelli-Itzykson-Zuber [27]. The trivial modular invariant $M = I$ exists for all levels k; a simple-current invariant $M[J]$ (see (2.9) below) exists for all even k; and there are exceptionals at $k = 10, 16, 28$. For instance, the level 28 exceptional is

$$\mathcal{Z}_{28}(q) = |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2$$

Cappelli-Itzykson-Zuber noticed something remarkable about their list: it falls into the A-D-E pattern. Each of their modular invariants M can be identified with the Dynkin diagram $\mathcal{G}(M)$ of a finite-dimensional simply laced Lie algebra (these are the diagrams A_n, D_n, E_n in Figure 1). The level of the modular invariant, plus 2, equals the Coxeter number h of $\mathcal{G}(M)$. Each number $1 \le a \le k+1$ will be an *exponent* of $\mathcal{G}(M)$ with multiplicity given by the diagonal entry $M_{a-1,a-1}$. The Coxeter number h and exponents m_i of the diagram $\mathcal{G}$ are listed in Table 1. For instance, the modular invariant $\mathcal{Z}_{28}$ given above corresponds to the E_8 Dynkin diagram: $28+2=30$, the Coxeter number of E_8; and the nonzero diagonal entries M_{bb} of M are at $b = 0, 6, 10, 12, 16, 18, 22, 28$, compared with the E_8 exponents $1, 7, 11, 13, 17, 19, 23, 29$ (all multiplicities being 1). Likewise, the D_8 Dynkin diagram has Coxeter number 14, and exponents $1, 3, 5, 7, 7, 9, 11, 13$, and corresponds to the $\mathrm{sl}(2)_{12}$ modular invariant

$$|\chi_0 + \chi_{12}|^2 + |\chi_2 + \chi_{10}|^2 + |\chi_4 + \chi_8|^2 + 2|\chi_6|^2$$

Table 1. Eigenvalues of graphs in Figure 1

Graph	Coxeter number h	Exponents m_i
A_n, $n \ge 1$	$n+1$	$1, 2, \ldots, n$
D_n, $n \ge 4$	$2n-2$	$1, 3, \ldots, 2n-3, n-1$
E_6	12	$1, 4, 5, 7, 8, 11$
E_7	18	$1, 5, 7, 9, 11, 13, 17$
E_8	30	$1, 7, 11, 13, 17, 19, 23, 29$
T_n, $n \ge 1$	$2n+1$	$1, 3, 5, \ldots, 2n-1$

Because of that observation, [7,16] suggested the following general definition.

Definition 2. *By the* exponents *of a modular invariant M, we mean the multi-set $\mathcal{E}_M$ consisting of $M_{\lambda\lambda}$ copies of λ for each $\lambda \in P_+$.*

(By a 'multi-set', we mean a set together with multiplicities, so $\mathcal{E}_M \subset P_+ \times \mathbb{Z}_{\ge}$.) In other words, the exponents are precisely the spin-0 primary fields in the theory (periodic sector). By analogy with the A-D-E classification for $\widehat{\mathrm{sl}}(2)$, we would like to assign graphs to a modular invariant in such a way that the eigenvalues of the graph (that is to say, the eigenvalues of its adjacency matrix) can be identified with the exponents of the modular invariant. We shall see next section that there is a natural way to do this: NIM-reps!

For example, $M = I$ has exponents $\mathcal{E}_I = P_+$, while the exponents $\mathcal{E}_C$ of $M = C$ are the self-conjugate primaries $\lambda = C\lambda$. In both $\mathcal{E}_I$ and $\mathcal{E}_C$, all multiplicities are 1, but simple-current invariants (see (2.9) below) can have arbitrarily large multiplicities.

It is merely a matter of convention whether $M_{\lambda,C\lambda} \ne 0$ or $M_{\lambda\lambda} \ne 0$ is taken as the definition of exponents — it has to do with the arbitrary choice of taking the holomorphic and antiholomorphic (i.e. left-moving and right-moving) chiral algebras to be isomorphic or anti-isomorphic. In the literature both choices can be found. We've taken them to be anti-isomorphic, hence our definition of spin-0 primaries.

Implicit in this discussion is the *diagonal* (i.e. identity) choice of 'gluing automorphism' Ω [28] or 'pairing' ω [5]. The pairing can be any permutation of P_+ which preserves fusions and conformal weights, or equivalently it can be any 'automorphism invariant', i.e. any modular invariant M which is a permutation matrix: $M_{\lambda\mu} = \delta_{\mu,\omega\lambda}$. For the current algebras, all possible pairings are given in [24]. This pairing tells one how to identify left- and right-moving primaries. Definition 2 can now be generalised to the multi-set $\mathcal{E}_M^\omega$, where λ occurs with multiplicity $M_{\lambda,\omega\lambda}$.

In this paper we will limit ourselves to the trivial (=identity) pairing ω. This is permitted for two reasons. First and most important, $\mathcal{E}_M^\omega = \mathcal{E}_{M\omega^t}$, where $M\omega^t$ is the modular invariant obtained by matrix multiplication. Second and quite intriguing, in practice it appears that the question of whether or not M is NIM-less (see §3.2 below) is independent of ω.

Consider a simple-current J with order n (so $J^n = id.$). Then we can find an integer R obeying $T_{J0,J0}T^*_{0,0} = \exp[2\pi\mathrm{i}\, R\,\frac{n-1}{2n}]$. Define a matrix $M[J]$ by [22]

$$M[J]_{\lambda\mu} = \sum_{\ell=1}^{n} \delta_{J^\ell\lambda,\mu}\,\delta(Q_J(\lambda) + \frac{\ell}{2n}R) \tag{2.9}$$

where $\delta(x) = 1$ if $x \in \mathbb{Z}$ and 0 otherwise. For example, $M[id.] = I$. The matrix $M[J]$ will be a modular invariant iff $T_{J0,J0}T^*_{0,0}$ is an nth root of 1 (this is automatic if n is odd; for n even, it's true iff R is even); it will in addition be a permutation matrix iff $T_{J0,J0}T^*_{0,0}$ has order exactly n. For example, for $\mathrm{sl}(2)_k$, $R = k$ so $M[J]$ is a modular invariant iff k is even, and when $k/2$ is odd it will in fact be a permutation matrix. The modular invariant $M[J]$ for $\mathrm{sl}(2)_{12}$ is given above.

We call these modular invariants *simple-current invariants*. This construction can be generalised somewhat when the simple-current group isn't cyclic, but (2.9) is good enough here. For any current algebra, at any sufficiently large level k, it appears that the only modular invariants are simple-current invariants and their product with C (except for the algebras $\mathrm{so}(4n)$, whose Dynkin symmetries permit (2.9) to be slightly generalised, and which have other 'conjugations' $C_i \neq C$).

We'll end this subsection by establishing some of the basic symmetries of modular invariants and their exponents. First, note that $MC = CM$ (since $C = S^2$), so λ and $C\lambda$ always appear in $\mathcal{E}_M$ with equal multiplicity. More generally, the Galois symmetry (2.6) of modular data yields an important modular invariant symmetry [23]:

$$M_{\lambda\mu} = M_{\sigma(\lambda),\sigma(\mu)} \tag{2.10a}$$

$$M_{\lambda\mu} \neq 0 \qquad \Longrightarrow \qquad \epsilon_\sigma(\lambda) = \epsilon_\sigma(\mu) \tag{2.10b}$$

for any Galois automorphism σ, i.e. any ℓ coprime to N. One thing (2.10a) implies is that λ and $\sigma(\lambda)$ will always have the same multiplicity in $\mathcal{E}_M$. This is quite strong — for instance, the primaries $0, 6, 10, 12, 16, 18, 22, 28$ for $\mathrm{sl}(2)_{28}$ all lie in the same Galois orbit, and indeed they all have the same multiplicity as exponents of the $k = 28$ exceptional modular invariant (just as they must for the other two $k = 28$ modular invariants).

The other fundamental symmetry of modular data is due to simple-currents. Let J, J' be simple-currents, and suppose that $M_{J0,J'0} \neq 0$. Then (see e.g. [18]) $\forall \lambda, \mu \in P_+$

$$M_{J\lambda,J'\mu} = M_{\lambda,\mu} \tag{2.11a}$$

$$M_{\lambda,\mu} \neq 0 \qquad \Longrightarrow \qquad Q_J(\lambda) \equiv Q_{J'}(\mu) \pmod 1 \tag{2.11b}$$

Thus by (2.11a), $J \in \mathcal{E}_M$ implies that all powers J^i are in $\mathcal{E}_M$, all with multiplicity 1, and also that λ and $J\lambda$ have the same multiplicity in $\mathcal{E}_M$ for any $\lambda \in P_+$.

It is curious that the selection rules (2.10b) and (2.11b) seem to have no direct consequences for $\mathcal{E}_M$, although they are profoundly important in constraining off-diagonal entries of M.

For later comparison, let's collect some of the main results on the exponents of modular invariants:

Theorem 1. Choose any modular data. Let M be any modular invariant, and let $\mathcal{E}_M$ be its exponents, with m_μ being the multiplicity $M_{\mu\mu}$ in $\mathcal{E}_M$.

(i) There are only finitely many modular invariants for that choice of modular data. They obey the bound $M_{\lambda\mu} \leq \frac{S_{\lambda 0}}{S_{00}} \frac{S_{\mu 0}}{S_{00}}$.

(ii) $m_0 = 1$.

(iii) For any simple-current J, $m_J = 0$ or 1; if $m_J = 1$ then $m_{J\lambda} = m_\lambda$ for all $\lambda \in P_+$.

(iv) For any Galois automorphism σ and any primary $\lambda \in P_+$, $m_{\sigma(\lambda)} = m_\lambda$.

(v) For any symmetry π of the fusion coefficients, and any $\lambda \in P_+$, we get

$$\sum_{\mu \in \mathcal{E}_{M\pi}} \frac{S_{\lambda\mu}}{S_{0\mu}} \in \mathbb{Z}_{\geq}$$

In (v) we sum over $\mathcal{E}_M$ as a multi-set, i.e. each μ appears m_μ times. The sum in (v) will typically be very large. By a symmetry of the fusion coefficients, we mean a permutation π of P_+ for which $N_{\lambda,\mu}^{\nu} = N_{\pi\lambda,\pi\mu}^{\pi\nu}$ for all $\lambda, \mu, \nu \in P_+$. It is equivalent to the existence of a permutation π' for which $S_{\pi\lambda,\pi'\mu} = S_{\lambda,\mu}$ — all such symmetries for the current algebras were found in [29]. To prove (v), let Π and Π' be the corresponding permutation matrices. Then

$$\sum_{\mu \in \mathcal{E}_{M\pi}} \frac{S_{\lambda,\mu}}{S_{0,\mu}} = \mathrm{Tr}(M\Pi D_\lambda) = \mathrm{Tr}(S^* S M \Pi D_\lambda) = \mathrm{Tr}(M\Pi' S D_\lambda S^*) = \mathrm{Tr}(M\Pi' N_\lambda) \in \mathbb{Z}_{\geq}$$

where D_λ is the diagonal matrix with entries $S_{\lambda\mu}/S_{0\mu}$. Thm.1(v) seems to be new.

Thm.1 assumes all $S_{\lambda 0} > 0$. If instead we have a *nonunitary* RCFT, let $o \in P_+$ be as in §2.1. Then we can show $m_o \geq 1$. However the known proofs that there are finitely many modular invariants, all break down, as does the proof of (iii).

In §3.3 as well as paragraph **(4)** in §8, we are interested in when simple-currents are exponents. Consider any matrix M which commutes with the T of $\mathrm{sl}(2)_k$. That is,

$$M_{ab} \neq 0 \qquad \Rightarrow \qquad (a+1)^2 \equiv (b+1)^2 \pmod{4(k+2)}$$

Thus a is odd iff b is odd, i.e. $Q_J(a) \equiv Q_J(b) \pmod 1$, provided $M_{ab} \neq 0$. If M is in addition a modular invariant, we get from this that

$$M_{J,J} = \sum_{a,b=0}^{k} S_{J,a}\, M_{ab}\, S^*_{J,b} = M_{00} = 1$$

Thus it is automatic for $\mathrm{sl}(2)_k$ that $J \in \mathcal{E}_M$, for all modular invariants M.

This argument generalises considerably. The norms of the weights of $\mathrm{sl}(n)_k$ satisfy

$$(\lambda|\lambda) \equiv Q_J(\lambda) - Q_J(\lambda)^2/n \pmod 2 \tag{2.12a}$$

where $Q_J(\lambda) = \sum_i i\lambda_i$, for the simple-current $J = k\Lambda_1$. For the basic calculation consider $\mathrm{sl}(3)_k$. Then commutation of M with T implies from (2.12a) the selection rule

$$M_{\lambda\mu} \neq 0 \qquad \Rightarrow \qquad Q_J(\lambda)^2 \equiv Q_J(\mu)^2 \pmod 3 \tag{2.12b}$$

and hence

$$\begin{aligned} M_{J,J} + M_{J,J^{-1}} &= \sum_{\lambda,\mu\in P_+} \left(\exp[2\pi\mathrm{i}\,\frac{Q_J(\lambda)-Q_J(\mu)}{3}] + \exp[2\pi\mathrm{i}\,\frac{Q_J(\lambda)+Q_J(\mu)}{3}]\right) S_{0\lambda} M_{\lambda\mu} S_{0\mu} \\ &= \sum_{\lambda,\mu\in P_+} \left(\cos[2\pi\frac{Q_J(\lambda)-Q_J(\mu)}{3}] + \cos[2\pi\frac{Q_J(\lambda)+Q_J(\mu)}{3}]\right) S_{0\lambda} M_{\lambda\mu} S_{0\mu} \end{aligned} \tag{2.12c}$$

where we use the reality of the LHS of (2.12c). But every term on the RHS of (2.12c) will be nonnegative: whenever $M_{\lambda\mu} \neq 0$, (2.12b) says that the sum of cosines in (2.12c) will either be $\frac{1}{2}$ or 2. Thus (2.12c) will be positive, so either $M_{J,J} \neq 0$ or $M_{J,J^{-1}} \neq 0$. In other words, for any $\mathrm{sl}(3)_k$ modular invariant M, either $J \in \mathcal{E}_M$ or $J \in \mathcal{E}_{MC} = \mathcal{E}_M^C$.

What we find, in this way, for an arbitrary current algebra, is:

Proposition 2. Let M be a modular invariant for some current algebra $X_{r,k}$ and let $\mathcal{E}_M$ and $\mathcal{E}_M^C = \mathcal{E}_{MC}$ be the sets of exponents, where C is charge-conjugation (2.1).

(i) For any $\mathrm{sl}(2)_k$, $\mathrm{so}(2n+1)_k = B_{n,k}$, $\mathrm{sp}(2n)_k = C_{n,k}$, and $E_{7,k}$, we have $J \in \mathcal{E}_M$.

(ii) For $\mathrm{sl}(n)_k = A_{n-1,k}$ when $n < 8$, as well as $E_{6,k}$, either $J \in \mathcal{E}_M$ or $J \in \mathcal{E}_M^C$.

(iii) More generally, for $\mathrm{sl}(n)_k = A_{n-1,k}$, define $n' = n$ or $n/2$ when n is odd or even, resp., and similarly $k' = k$ or $k/2$ when k is odd or even, resp. Define a_i by the prime decomposition $n' = \prod_i p_i^{a_i}$, and let $s = \prod_i p_i^{\lfloor a_i/2 \rfloor}$. Assume $\gcd(n', k')$ equals 1 or a power of a single prime. Then there exists an automorphism invariant (='pairing') ω such that the simple-current $J^s = k\Lambda_s$ lies in $\mathcal{E}_M^\omega = \mathcal{E}_{M\omega^t}$.

(iv) For $\mathrm{so}(2n)_k = D_{n,k}$, when 4 doesn't divide n, the simple-current $J_v = k\Lambda_1$ lies in $\mathcal{E}_M$.

The simple-current J in (i)-(iii) is any generator of the corresponding (cyclic) groups of simple-currents. By '$\lfloor a_i/2 \rfloor$' here we mean to truncate to the nearest integer not greater than $a_i/2$. Note that s is the largest number such that s^2 divides n or (if n is even) $n/2$. For instance $s = 1, 2, 3$ for $n = 4, 8, 18$. The automorphism invariants π for $\mathrm{sl}(n)_k$ are explicitly given in [24]. To prove (iii), use (2.12) to obtain $M_{J^s,J^{s\ell}} = 1$ for some integer

ℓ congruent to ± 1 modulo an appropriate power of each prime p_i; the automorphism invariants ω (when they exist) can be seen to reverse those signs.

When instead distinct primes divide $\gcd(n', k')$, a multiple s' of s will work in (iii): namely, choose any prime (say p_1) dividing both n' and k', and define $s' = p_1^{\lfloor a_1 \rfloor} \prod_i p_i^{\lfloor a_i/2 \rfloor} \prod_j p_j^{a_j}$ where the p_i don't divide k', and the p_j ($j \neq 1$) do.

2.3. Quick review of matrix and graph theory.

We will write A^t for the transpose of A. By a $\mathbb{Z}_{\geq}$-matrix we mean a matrix whose entries are nonnegative integers. Two $n \times n$ matrices A and B are called *equivalent* if there is a permutation which, when applied simultaneously to the rows and columns of A, yields B — i.e. $B = \Pi A \Pi^t$. The direct sum $A \oplus B$ of an $n \times n$ matrix A and $m \times m$ matrix B is the $(n+m) \times (n+m)$ matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. A matrix M is called *decomposible* if it can be written in the form (i.e. is equivalent to) $A \oplus B$, otherwise it is called *indecomposible.* A matrix N is called *reducible* if it is equivalent to a matrix of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ for submatrices A, B, C where $B \neq 0$. Fortunately, all of our matrices turn out to be irreducible.

For example, two $n \times n$ permutation matrices Π and Π' are equivalent iff the corresponding permutations π and π' are conjugate in the symmetric group S_n (i.e. have the same numbers of disjoint 1-cycles, 2-cycles, 3-cycles, etc). They will be indecomposible iff they are transitive, i.e. iff they are equivalent to the $n \times n$ matrix

$$\Pi^{(n)} := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{2.13}$$

in which case they will also be irreducible.

The eigentheory (i.e. the Perron-Frobenius theory — see e.g. [30]) of nonnegative matrices is fundamental to the study of NIM-reps. The basic result is that if A is a square matrix with nonnegative entries, then there is an eigenvector with nonnegative entries whose eigenvalue $r(A)$ is nonnegative. The eigenvector(resp., -value) is called the *Perron-Frobenius eigenvector(-value).* This eigenvalue has the additional property that if s is any other eigenvalue of A, then $r(A) \geq |s|$. There are many other results in Perron-Frobenius theory that we'll use, but we'll recall them as needed.

The matrices with small $r(A)$ have been classified (see especially [31] for $r(A) < \sqrt{2+\sqrt{5}} \approx 2.058$), but unfortunately with a weaker notion of 'equivalence' than we would like. The moral of the story seems to be that these matrix classifications are very difficult, and will be hopeless as $r(A)$ gets much larger than 2; the only hope is to simultaneously impose other conditions on the matrix, e.g. some symmetries. Fortunately, we can always find other conditions obeyed by our matrices, besides the value of r.

This is one of the places where 'irreducibility' simplifies things. For instance, $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $\forall k$, are some of the indecomposable $\mathbb{Z}_{\geq}$-matrices with maximum eigenvalue $r(A) = 1$, but the only 2×2 indecomposable *irreducible* $\mathbb{Z}_{\geq}$-matrix with $r(A) = 1$ is $\Pi^{(2)}$.

An irreducible $\mathbb{Z}_{\geq}$-matrix will have at most $r(A)^2$ nonzero entries in each row, and so for small $r(A)$ will be quite sparse. A sparse matrix is usually more conveniently depicted as a *(multi-di)graph*. For example, in Lie theory a Dynkin diagram replaces the Cartan matrix. The same trick is used here, and is responsible for the beautiful pictures scattered throughout the NIM-rep literature (see e.g. [7,9]).

By a *graph* we allow loops (i.e. an edge starting and ending at the same vertex), but its edges aren't directed and aren't multiple. A multi-digraph is the generalisation which allows multiple edges and directed edges. We assign a multi-digraph to a $\mathbb{Z}_{\geq}$-matrix A as follows. For any i, j, draw A_{ij} edges directed from vertex i to vertex j. Replace each pair $i \to j, j \to i$ of directed edges with an undirected one connecting i and j (except we never put arrows on loops).

We are very interested in the spectra of (multi-di)graphs, i.e. the list of eigenvalues (with multiplicities) of the associated adjacency matrix. There has been a lot of work on this in recent years — see e.g. the readable book [32]. We will state the results as we need them. A major lesson from the theory: the eigenvalues usually won't determine the graph. For example, the graphs $D_4^{(1)}$ and $A_3^{(1)} \cup A_1$ have identical spectra.

There are no nonzero irreducible $\mathbb{Z}_{\geq}$-matrices with $r(A) < 1$. The only $n \times n$ irreducible indecomposable $\mathbb{Z}_{\geq}$-matrix with $r(A) = 1$ is $\Pi^{(n)}$ in (2.13), up to equivalence. The connected graphs $\mathcal{G}$ with $r(\mathcal{G}) < 2$ or $r(\mathcal{G}) = 2$ — i.e. symmetric indecomposable matrices with $r(A) < 2$ or $r(A) = 2$ — are given in Figures 1 and 2, and the loop-less ones (=traceless matrices) fall into the famous A-D-E pattern (in fact these seem to be the two prototypical A-D-E patterns). Incidentally, Tables 1 and 2 give all the eigenvalues of the graphs in Figures 1 and 2, respectively. For Figure 1 these eigenvalues are $2\cos(\pi m_i/h)$.

Table 2. Eigenvalues of graphs in Figure 2

Graph	eigenvalues	range
$A_n^{(1)}$, $n \geq 1$	$2\cos(2\pi k/(n+1))$	$0 \leq k \leq n$
$D_n^{(1)}$, $n \geq 4$	$0, 0, 2\cos(\pi k/(n-2))$	$0 \leq k \leq n-2$
$E_6^{(1)}$	$\pm 2, \pm 1, \pm 1, 0$	
$E_7^{(1)}$	$\pm 2, \pm\sqrt{2}, \pm 1, 0, 0$	
$E_8^{(1)}$	$\pm 2, \pm 2\cos(\pi/5), \pm 1, \pm 2\cos(2\pi/5), 0$	
${}^0A_n^0$, $n \geq 1$	$2\cos(k\pi/n)$	$0 \leq k < n$
D_n^0, $n \geq 3$	$0, 2\cos(2\pi k/(2n-3))$	$0 \leq k \leq n-2$

Let $\mathcal{G}$ be any multi-digraph. We'll write $r(\mathcal{G})$ for the Perron-Frobenius eigenvalue of its adjacency matrix. $\mathcal{G}$ is called *bipartite* if its vertices can be coloured black and white, in such a way that the endpoints of any (directed) edge are coloured differently. For example, any tree is bipartite. If $\mathcal{G}$ is connected and its adjacency matrix is irreducible, it will be bipartite iff the number $-r(\mathcal{G})$ is also an eigenvalue of $\mathcal{G}$.

3. NIM-reps

3.1. The physics of NIM-reps.

In this section we introduce the main subject of the paper: NIM-reps. Recall the discussion in the Introduction. Fix an RCFT and a choice of chiral algebra. In other words, we are given modular data S and T and a modular invariant M. We are interested here in boundary conditions which are not only conformally invariant, but also don't break the given chiral algebra.

Let $x \in \mathcal{B}$ parametrise the $\mathbb{Z}_{\geq}$-basis for the boundary states in the RCFT (or Chan-Paton degrees-of-freedom for an open string). Consider the 1-loop vacuum-to-vacuum amplitude of an open string, i.e. the amplitude associated to a cylinder whose boundaries are labelled with states $|x\rangle, |y\rangle$. Then we can write them as

$$\mathcal{Z}_{x|y}(q) = \sum_{\lambda \in P_+} \mathcal{N}_{\lambda x}^{y}\, \chi_\lambda(q) \tag{3.1a}$$

where $\mathcal{N}_{\lambda x}^{y} \in \mathbb{Z}_{\geq}$ and $\chi_\lambda(q)$ are the usual RCFT (e.g. current algebra) characters. The parameter $0 < q < 1$ parametrises the conformal equivalence classes of cylinders, just as $|q| < 1$ did for tori in (2.8). Depending on how we choose the time direction, we can interpret the cylinder either as a 1-loop open string worldsheet, or a 0-loop closed string worldsheet; using this Cardy [1] derived (at least for $M = I$)

$$\mathcal{Z}_{x|y}(q) = \sum_{\lambda \in P_+} \sum_{\mu \in \mathcal{E}} \frac{U_{x\mu}\, S_{\lambda\mu}\, U_{y\mu}^*}{S_{0\mu}}\, \chi_\lambda(q) \tag{3.1b}$$

Here $\mathcal{E}$ is the exponents $\mathcal{E}_M$ of the modular invariant M for the RCFT. The matrix entries $U_{x\mu}$ (appropriately normalised) give the change-of-coordinates from boundary states $|x\rangle$, $x \in \mathcal{B}$, to the Ishibashi states $|\mu\rangle\!\rangle$, $\mu \in \mathcal{E}_M$. The matrices $\mathcal{N}_\lambda$ given by

$$(\mathcal{N}_\lambda)_{xy} = \mathcal{N}_{\lambda x}^{y} = \sum_{\mu \in \mathcal{E}} \frac{U_{x\mu}\, S_{\lambda\mu}\, U_{y\mu}^*}{S_{0\mu}} \tag{3.1c}$$

constitute what we will soon call a NIM-rep. Note by taking complex conjugation of (3.1c) that $\mathcal{N}_\lambda^t = \mathcal{N}_{C\lambda}$. We will require that U be unitary (although the physical reasons for this are not so clear). This gives us (3.2a) below.

3.2. Basic definitions.

Definition 3. *By a* NIM-rep $\mathcal{N}$ *we mean an assignment of a matrix $\mathcal{N}_\lambda$, with nonnegative integer entries, to each $\lambda \in P_+$ such that $\mathcal{N}$ forms a representation of the fusion ring:*

$$\mathcal{N}_\lambda\, \mathcal{N}_\mu = \sum_{\nu \in P_+} N_{\lambda\mu}^{\nu}\, \mathcal{N}_\nu \tag{3.2a}$$

for all primaries $\lambda, \mu, \nu \in P_+$, and also that

$$\mathcal{N}_0 = I \tag{3.2b}$$

$$\mathcal{N}_{C\lambda} = \mathcal{N}_\lambda^t \qquad \lambda \in P_+ \tag{3.2c}$$

The NIM-rep '$\mathcal{N}$' should not be confused with the fusion 'N'. In (3.2c), 'C' denotes charge-conjugation (2.1), and 't' denotes transpose. Equation (3.2b) isn't significant, and serves to eliminate from consideration the trivial $\lambda \mapsto (0)$. Further refinements of Definition 3 are probably desirable, and are discussed in paragraphs **(4)**,**(5)** in §8.

The *dimension* n of a NIM-rep is the size $n \times n$ of the matrices $\mathcal{N}_\lambda$. Note that our definition is more general (i.e. fewer conditions) than in older papers by (various subsets of) Di Francesco&Petkova&Zuber. The *fusion graphs* of $\mathcal{N}$ are the multi-digraphs associated to the matrices $\mathcal{N}_\lambda$.

We call two n-dimensional NIM-reps $\mathcal{N}, \mathcal{N}'$ *equivalent* if there is an $n \times n$ permutation matrix P (independent of $\lambda \in P_+$) such that $\mathcal{N}'_\lambda = P\mathcal{N}_\lambda P^{-1}$ for all $\lambda \in P_+$. Obviously we can and should identify NIM-reps equivalent in this sense. More generally, when that same relation holds for some arbitrary invertible (i.e. not necessarily permutation) matrix P, we will call $\mathcal{N}$ and $\mathcal{N}'$ *linearly equivalent.* At least 3 distinct NIM-reps for sl(3)$_9$ have been found with identical exponents [7], which shows that linear equivalence is strictly weaker than equivalence (similar examples are known [11] for sl(4)$_8$). In fact, linear equivalence isn't important, and doesn't respect the physics.

One way to build new NIM-reps from old ones $\mathcal{N}', \mathcal{N}''$ is *direct sum* $\mathcal{N} = \mathcal{N}' \oplus \mathcal{N}''$:

$$\mathcal{N}_\lambda := \mathcal{N}'_\lambda \oplus \mathcal{N}''_\lambda = \begin{pmatrix} \mathcal{N}'_\lambda & 0 \\ 0 & \mathcal{N}''_\lambda \end{pmatrix}$$

We call such a representation $\mathcal{N}$ *decomposable* (or *reducible*); $\mathcal{N}$ is *indecomposable* when the $\mathcal{N}_\lambda$'s cannot be simultaneously put into block form. Obviously, an arbitrary NIM-rep can always be written as (i.e. is equivalent to) a direct sum of indecomposable NIM-reps, so it suffices to consider the indecomposable ones. Physically, decomposable NIM-reps would correspond to completely decoupled blocks of boundary conditions. We will show in §3.3 that for any given choice of modular data, there are only finitely many indecomposable NIM-reps $\mathcal{N}$.

Two obvious examples of NIM-reps are given by fusion matrices, namely $\mathcal{N}_\lambda = N_\lambda$, and $\mathcal{N}_\lambda = N_\lambda^t$. Both are indecomposable, but they are equivalent: in fact, $N_\lambda^t = CN_\lambda C^{-1}$. We call this obvious NIM-rep the *regular* one.

The matrices $\mathcal{N}_\lambda$ of §3.1 define a NIM-rep. Thus to any RCFT should correspond a NIM-rep, and it should play as fundamental a role as the modular invariant.

Let $\mathcal{N}$ be any NIM-rep. Equation (3.2a) tells us that the matrices $\mathcal{N}_\lambda$ pairwise commute; (3.2c) then tells us that they are normal. Thus they can be simultaneously diagonalised, by a unitary matrix U. Each eigenvalue $e_\lambda(a)$ defines a 1-dimensional representation $\lambda \mapsto e_\lambda(a)$ of the fusion ring, so $e_\lambda(a) = \frac{S_{\lambda\mu}}{S_{0\mu}}$ for some $\mu \in P_+$. Thus any NIM-rep will necessarily obey the Verlinde-like decomposition (3.1c), for some multi-set $\mathcal{E} = \mathcal{E}(\mathcal{N})$. We will call $\mathcal{E}$ the *exponents* of the NIM-rep, by analogy with the A-D-E classification of $\widehat{\mathrm{sl}}(2)$. Note that $\mathcal{N}$ and $\mathcal{N}'$ are linearly equivalent iff their exponents $\mathcal{E}(\mathcal{N}),\mathcal{E}(\mathcal{N}')$ are equal (including multiplicities).

At first glance, there doesn't seem to be much connection between NIM-reps and modular invariants. But the discussion in §1 tells us that the NIM-rep $\mathcal{N}$ and modular invariant M of an RCFT should obey the compatibility relation

$$\mathcal{E}(\mathcal{N})=\mathcal{E}_M \tag{3.3}$$

Thus we want to pair up the NIM-reps with the modular invariants so that (3.3) is satisfied; any NIM-rep (resp. modular invariant) without a corresponding modular invariant (resp. NIM-rep) can be regarded as having questionable physical merit (more precisely, before a modular invariant is so labelled, all possible pairings ω should be checked — see §2.2).

Definition 4. *We call a modular invariant M* NIMmed *if there exists a NIM-rep $\mathcal{N}$ compatible with M in the sense of* (3.3)*. Otherwise we call M* NIM-less.

For instance the regular NIM-rep $\mathcal{N}_\lambda=N_\lambda$ has exponents $\mathcal{E}=P_+$, as does the modular invariant $M=I$. Thus they are paired up. It is easy to verify that the only modular invariant M with $\mathcal{E}_M\supseteq P_+$ is $M=I$, so it is the unique modular invariant which can be paired with the regular NIM-rep. It would be interesting to find other indecomposable NIM-reps with $\mathcal{E}(\mathcal{N})\supseteq P_+$. The Cardy ansatz [1] is essentially the statement that $\mathcal{E}(\mathcal{N})=P_+$ implies $\mathcal{N}$ is the regular NIM-rep.

Suppose the RCFT has a discrete symmetry G. We can consider fields in the theory with twisted, nonperiodic boundary conditions induced by the action of G. The resulting partition functions $\mathcal{Z}_{g,g'}(\tau)$ (one for each twisted sector of the theory) are *sub*modular invariants. The philosophy of [33] is that what one can do (e.g. study NIM-reps) with the modular invariant $\mathcal{Z}_{e,e}$, can be done as well for the submodular invariants $\mathcal{Z}_{g,g'}$ — indeed this is a way of probing the global structure of the theory. The material of this paper, e.g. the Thm.1↔Thm.3 correspondence, should be generalised to this more general situation.

Let $\Gamma=\{\gamma^{(1)},\dots,\gamma^{(g)}\}$ be any fusion-generator of P_+. From (3.1c) and (2.7) it is easy to see that

$$\mathcal{N}_\lambda=P_\lambda(\mathcal{N}_{\gamma^{(1)}},\dots,\mathcal{N}_{\gamma^{(g)}})\qquad\forall\lambda\in P_+ \tag{3.4}$$

for any NIM-rep $\mathcal{N}$. Thus for $\widehat{\mathrm{sl}}(2)$ and $\widehat{\mathrm{sl}}(3)$ NIM-reps, the entire $\mathcal{N}$ is uniquely determined by knowing the first-fundamental $\mathcal{N}_{\Lambda_1}$, or equivalently its fusion graph. But for most choices of $\mathrm{sl}(n)_k$ (see §2.1 for the complete answer), knowing $\mathcal{N}_{\Lambda_1}$ is not enough to determine all of $\mathcal{N}$.

Several fusion graphs for $\widehat{\mathrm{sl}}(3)$ are given in [7]. They make no claims for the completeness of their lists, and in fact it is not hard to find missing ones. To give the simplest example, the 1-dimensional $\mathrm{sl}(3)_3$ NIM-rep (given by the quantum-dimension 1, 2 or 3) is missing. Incidentally, 1-dimensional $\mathrm{sl}(n)_k$ NIM-reps exist only for $\mathrm{sl}(n)_1$, $\mathrm{sl}(2)_2$, $\mathrm{sl}(2)_4$, $\mathrm{sl}(3)_3$, and $\mathrm{sl}(4)_2$ (for a proof, see p.691 of [34]).

3.3. The basic theory of NIM-reps.

This section is central to the paper. Most of the results here are new. For simple examples using them, see §§6,7. Although we go much further, some consequences were already obtained in especially [35], using more restrictive axioms.

Let $\mathcal{N}$ be any indecomposable NIM-rep. Let $\mathcal{E}(\mathcal{N})$ be its exponents, and for any exponent $\mu \in \mathcal{E}(\mathcal{N})$, let m_μ denote the multiplicity. Then the matrix $\sum_{\lambda\in P_+}\mathcal{N}_\lambda$ is strictly positive, and $m_0 = 1$. More generally, the value of m_0 tells you how many indecomposable summands $\mathcal{N}^i$ there are in a decomposable $\mathcal{N} = \oplus_i \mathcal{N}^i$.

To see this, write '$x \sim y$' if $\mathcal{N}_{\lambda x}^y \neq 0$ for some λ. Then this defines an equivalence relation on $\mathcal{B}$: $x \sim x$ because $\mathcal{N}_0 = I$; if $x \sim y$ then $y \sim x$, because $\mathcal{N}_{\lambda x}^y = \mathcal{N}_{C\lambda,y}^x$; if $x \sim y$ (say $\mathcal{N}_{\lambda x}^y \neq 0$) and $y \sim z$ (say $\mathcal{N}_{\mu y}^z \neq 0$) then $x \sim z$, because $(\mathcal{N}_\lambda \mathcal{N}_\mu)_{xz} \neq 0$. So we get a partition $\mathcal{B}_i$ of $\mathcal{B}$ such that $\sum_{\lambda\in P_+}\mathcal{N}_\lambda$ restricted to each $\mathcal{B}_i$ is strictly positive, but $(\sum_{\lambda\in P_+}\mathcal{N}_\lambda)_{xy} = 0$ when $x \in \mathcal{B}_i, y \in \mathcal{B}_j$ belong to different classes. This tells us that $\mathcal{N}$ is the direct sum of the $\mathcal{N}^{(i)}$ (the restriction of $\mathcal{N}$ to the subset $\mathcal{B}_i$), so $\mathcal{N}$ being indecomposable requires that there be only one class $\mathcal{B}_i$ (i.e. that $\mathcal{B}_i = \mathcal{B}$). The reason this forces $m_0 = 1$ is because of Perron-Frobenius theory [30]: the multiplicity of the Perron-Frobenius eigenvalue for a strictly positive matrix (e.g. $\sum_{\lambda\in P_+}\mathcal{N}_\lambda$) is 1.

Consider $\mathcal{N}$ indecomposable. The Perron-Frobenius eigenspace of $\sum_\lambda \mathcal{N}_\lambda$ will then be one-dimensional, spanned by a strictly positive vector $\vec{v}$. Now the simultaneous eigenspaces of the matrices $\mathcal{N}_\lambda$ will necessarily be a partition of the eigenspaces of e.g. $\sum_\lambda \mathcal{N}_\lambda$. Thus $\vec{v}$ must be an eigenvector (hence a Perron-Frobenius eigenvector) of all $\mathcal{N}_\lambda$. Suppose $\vec{v}$ corresponds to exponent $\mu \in \mathcal{E}(\mathcal{N})$. Then its eigenvalues $S_{\lambda\mu}/S_{0\mu}$ must all be positive. The only primary $\mu \in P_+$ with this property for all $\lambda \in P_+$, is $\mu = 0$. This means that we know the Perron-Frobenius eigenvalue of any matrix $\mathcal{N}_\lambda$: it's simply the quantum-dimension

$$r(\mathcal{N}_\lambda) = \frac{S_{\lambda 0}}{S_{00}} \tag{3.5}$$

Let U be a unitary diagonalising matrix of all $\mathcal{N}_\lambda$, as in (3.1c) (its existence was proved last subsection). U will not be unique, but it can be chosen to have properties reminiscent of S. In particular the column $U_{\updownarrow 0}$ can be chosen to be the Perron-Frobenius eigenvector $\vec{v}$ (normalised), so each entry obeys $U_{x0} > 0$. We will discuss U in more detail in §3.4.

This argument also tells us the important fact that if the matrix $\mathcal{N}_\lambda$ is a direct sum of indecomposable matrices A_i, then each A_i (equivalently each component of the fusion graph of λ) must have the same maximal eigenvalue $r(A_i) = S_{\lambda 0}/S_{00}$. The reason is that $\mathcal{N}_\lambda$ has a strictly positive eigenvector, namely $\vec{v}$. Moreover, these matrices A_i will be *irreducible* (see §2.3 for the definition). This follows for example from Corollary 3.15 of [30] — in particular, the Perron-Frobenius vector for both $\mathcal{N}_\lambda$ and $\mathcal{N}_\lambda^t = \mathcal{N}_{C\lambda}$ is the vector $\vec{v} > 0$.

Clearly, all $\mathcal{N}_\lambda$ are symmetric iff all exponents $\mu \in \mathcal{E}$ satisfy $\mu = C\mu$. More generally,

$$\mathcal{N}_\lambda = \mathcal{N}_\nu \qquad \text{iff} \qquad S_{\lambda\mu} = S_{\nu\mu} \qquad \forall \mu \in \mathcal{E}(\mathcal{N}) \tag{3.6}$$

So for any simple-current J, $\mathcal{N}_J = I$ iff $Q_J(\mu) \in \mathbb{Z}$ $\forall \mu \in \mathcal{E}$, in which case $\mathcal{N}_{J\lambda} = \mathcal{N}_\lambda$ $\forall \lambda \in P_+$. More generally, by (3.5) $\mathcal{N}_J$ will be a permutation matrix. If we let j denote the permutation of the vertices $\mathcal{B}$, corresponding to $\mathcal{N}_J$, then the order of j will be the least common multiple of all the denominators of $Q_J(\mu)$ as μ runs over $\mathcal{E}$. Thus the order of j will always divide the order of J. Moreover,

$$(\mathcal{N}_{J\lambda})_x^y = \mathcal{N}_{\lambda,jx}^y = \mathcal{N}_{\lambda,x}^{j^{-1}y}$$

We will prove in Theorem 3 below the very useful and nontrivial facts that the multiplicity m_J of any simple-current must be 0 or 1, and if it is 1 then J will be a symmetry of $\mathcal{E}$ — i.e. $m_{J\mu} = m_\mu$ for all $\mu \in \mathcal{E}$. It follows from this that the simple-currents in $\mathcal{E}$ form a group, which we'll denote $\mathcal{E}_{sc}$.

Fix any vertex $1 \in \mathcal{B}$. By an $\mathcal{N}_1$-grading g we mean a colouring $g(x) \in \mathbb{Q}$ of the vertices $\mathcal{B}$ and colouring $g_\lambda \in \mathbb{Q}$ of the primaries P_+, such that $g(1) \in \mathbb{Z}$ and

$$\mathcal{N}_{\lambda x}^{y} \neq 0 \qquad \Rightarrow \qquad g_\lambda + g(x) \equiv g(y) \pmod 1 \tag{3.7}$$

Clearly the $\mathcal{N}_1$-gradings form a group under addition; different choices of '1' yield isomorphic groups. Thm.3(viii) says that this group is naturally isomorphic to the group of simple-currents in $\mathcal{E}$. In particular, to any simple-current $J \in \mathcal{E}$ we get an $\mathcal{N}_1$-grading as follows. Define $g_\lambda = Q_J(\lambda)$, and put $g(y) = Q_J(\lambda)$ if $\mathcal{N}_{\lambda x}^{y} \neq 0$ for some $\lambda \in P_+$. This defines an $\mathcal{N}_1$-grading, and we learn in Thm.3(viii) that all $\mathcal{N}_1$-gradings arise in this way.

Let A be any matrix, and let m_s be the multiplicity of eigenvalue s. If all entries of A are rational, then each eigenvalue s will be an algebraic number (since it's the root of a polynomial over $\mathbb{Q}$). If σ is any Galois automorphism (of the splitting field of the characteristic polynomial of A), and s is any eigenvalue, then the image $\sigma(s)$ will also be an eigenvalue of A, and the multiplicities m_s and $m_{\sigma(s)}$ will be equal.

Now, $\sigma\frac{S_{\lambda\mu}}{S_{0\mu}} = S_{\lambda,\sigma\mu}/S_{0,\sigma\mu}$, by (2.6). So what this means is that the multiplicities $m_\mu, m_{\sigma(\mu)}$ of μ and $\sigma(\mu)$ in the exponents $\mathcal{E}(\mathcal{N})$ must be equal — that is, the exponents $\mathcal{E}(\mathcal{N})$ obey the same Galois symmetry as the exponents $\mathcal{E}_M$ (see Thm.1(iv)).

A special case of this is that λ and $C\lambda$ have the same multiplicity. That follows from (is equivalent to) the fact that the entries $\mathcal{N}_{\lambda x}^{y}$ are all *real*. The much more general Galois symmetry follows from (and together with (3.10a) is equivalent to) the much stronger statement that each $\mathcal{N}_{\lambda x}^{y}$ is *rational*.

Note that for any $\lambda \in P_+$,

$$\mathrm{Tr}(\mathcal{N}_\lambda) = \sum_{\mu \in \mathcal{E}(\mathcal{N})} \frac{S_{\lambda\mu}}{S_{0\mu}} \in \mathbb{Z}_{\geq} \qquad \forall \lambda \in P_+ \tag{3.8}$$

This is a strong condition for a multi-set $\mathcal{E}$ to obey — see e.g. §7. If $\mathcal{E}$ obeys the Galois condition $m_\lambda = m_{\sigma(\lambda)}$, as it must, then the sum in (3.8) will automatically be integral, so the important thing in (3.8) is nonnegativity.

Suppose $\mathcal{N}$ is indecomposable. Then

$$\sum_{\lambda \in P_+} S_{0\lambda}\,\mathrm{Tr}(\mathcal{N}_\lambda) = \sum_{\lambda \in P_+}\sum_{\mu \in \mathcal{E}} S_{0\lambda}\,\frac{S_{\lambda\mu}}{S_{0\mu}} = \frac{1}{S_{00}} \tag{3.9a}$$

All LHS terms are nonnegative. By considering the contribution to the LHS by $\lambda = 0$, we find that the dimension of an indecomposable NIM-rep is bounded above by S_{00}^{-2}.

Moreover, each entry of $\mathcal{N}_\lambda$ must be bounded above by the quantum-dimension $S_{\lambda 0}/S_{00}$. To see this, note that the matrix $\mathcal{N}_\lambda \mathcal{N}_\lambda^t$ has largest eigenvalue $r = (S_{\lambda 0}/S_{00})^2$; by

Perron-Frobenius theory any diagonal entry A_{ii} of a nonnegative matrix A will be bounded above by $r(A)$. Thus for each i, j we get

$$((\mathcal{N}_\lambda)_{ij})^2 \leq (\mathcal{N}_\lambda \mathcal{N}_\lambda^t)_{ii} \leq (S_{\lambda 0}/S_{00})^2 \tag{3.9b}$$

Together, these two bounds tell us that *the number of indecomposable NIM-reps, for a fixed choice of modular data, must be finite.*

We collect next the main things we've established.

Theorem 3. Choose any modular data, and let $\mathcal{N}$ be any indecomposable NIM-rep, with exponents $\mathcal{E}$ and multiplicities m_λ.

(i) For the given modular data, there are only finitely many different indecomposable NIM-reps. We have the bounds $(\mathcal{N}_\lambda)_{ij} \leq S_{\lambda 0}/S_{00}$ and $\dim(\mathcal{N}) \leq S_{00}^{-2}$.

(ii) $m_0 = 1$.

(iii) For any simple-current J, either $m_J = 0$ or 1; if $m_J = 1$ then $m_{J\lambda} = m_\lambda$ for any primaries $\lambda \in P_+$.

(iv) For any Galois automorphism σ and primary $\lambda \in P_+$, $m_{\sigma(\lambda)} = m_\lambda$.

(v) For any primary $\lambda \in P_+$, inequality (3.8) holds.

(vi) For any primary $\lambda \in P_+$, each indecomposable submatrix of $\mathcal{N}_\lambda$ will be irreducible and have largest eigenvalue equal to the quantum-dimension $S_{\lambda 0}/S_{00}$ of λ. The number of indecomposable components will precisely equal the number of $\mu \in \mathcal{E}$ such that $S_{\lambda\mu}/S_{0\mu} = S_{\lambda 0}/S_{00}$. The number of these components which have a $\mathbb{Z}_m$-grading is precisely the number of $\mu \in \mathcal{E}$ with $S_{\lambda\mu}/S_{0\mu} = e^{2\pi \mathrm{i}/m}\, S_{\lambda 0}/S_{00}$.

(vii) No row or column of any matrix $\mathcal{N}_\lambda$ can be identically 0.

(viii) Fix any vertex $1 \in \mathcal{B}$. The $\mathcal{N}_1$-gradings of the NIM-rep are in a natural one-to-one correspondence with the simple-currents $J \in \mathcal{E}$.

(ix) Let $\mathcal{E}_{sc}$ denote the set of all simple-currents in $\mathcal{E}$, $\mathcal{S}_{sc}$ denote all simple-currents in P_+, and $\mathcal{S}_0$ be the set of all simple-currents $J \in P_+$ such that $Q_J(J') \in \mathbb{Z}$ for all $J' \in \mathcal{E}_{sc}$. Then $\|\mathcal{S}_{sc}\|$ must divide $\|\mathcal{S}_0\| \dim(\mathcal{N})$.

(x) If a primary $\lambda \in P_+$ has $Q_J(\lambda) \notin \mathbb{Z}$ for some simple-current $J \in \mathcal{E}$, then $\mathcal{N}_{\lambda x}^x = 0$ for all $x \in \mathcal{B}$.

Note that the grading in (vi) applies to an individual matrix $\mathcal{N}_\lambda$, whereas that of (viii) refers to a grading valid simultaneously for all matrices $\mathcal{N}_\lambda$. Part (vii) comes from applying nonnegativity to $(\mathcal{N}_\lambda)(\mathcal{N}_\lambda)^t = I + \cdots$. Part (x) comes from (3.8) and Thm.3(iii). The remainder of the proof of Thm.3 is relegated to the end of the appendix.

Compare Theorems 1 and 3: surprisingly, the general properties obeyed by the exponents of a modular invariant, and those of a NIM-rep, match remarkably well. It would be nice to obtain a simple, general, and effective test for the NIM-lessness of a modular invariant. One candidate is Thm.3(ix): this author has managed to show for modular invariants, only the weaker statement that $\|\mathcal{E}(M)_{sc}\|$ must divide $\|\mathcal{E}(M)_{sc} \cap \mathcal{S}_0\| \,\mathrm{Tr}(M)$, where $\mathcal{E}(M)_{sc}$ equals the number of simple-currents in $\mathcal{E}_M$.

Thm.3 assumes all $S_{\lambda 0} > 0$. For nonunitary RCFT, let $o \in P_+$ be as in §2.1. Then 3(ii) becomes $m_o = 1$, but m_0 seems unconstrained. The bound on $\dim(\mathcal{N})$ is now S_{0o}^{-2}. In 3(iii) replace m_J with m_{Jo}.

There are several generic constructions of NIM-reps, and a systematic study of these should probably be made. We will only mention one, which seems to have been overlooked in the literature. It involves the notion of *fusion-homomorphism*, i.e. a map $\pi : P_+ \to P'_+$ between the primaries of two (possibly identical) fusion rings, which defines a ring homomorphism of the corresponding fusion rings: that is,

$$\pi(\lambda) \boxtimes' \pi(\mu) = \sum_{\nu \in P_+} N_{\lambda\mu}^{\nu} \pi(\nu)$$

where $\boxtimes'$ is the fusion product for P'_+. See Prop.3 of [18] for its basic properties. In particular, there exists a map $\pi' : P'_+ \to P_+$ such that [18]

$$\frac{S'_{\pi\lambda,\mu'}}{S'_{0',\mu'}} = \frac{S_{\lambda,\pi'\mu'}}{S_{0,\pi'\mu'}}$$

Also, $\pi\lambda = \pi\mu$ iff $\mu = J\lambda$ for some simple-current J with $\pi(J) = 0$.

Suppose $\pi : P_+ \to P'_+$ is a fusion-homomorphism, and $\mathcal{N}$ is a NIM-rep of P'_+. Then $\mathcal{N}^\pi$ defined by $(\mathcal{N}^\pi)_\lambda = \mathcal{N}_{\pi\lambda}$ is a (usually decomposable) NIM-rep of P_+. For a trivial example, when π is a fusion-isomorphism, and $\lambda \mapsto N_\lambda$ is the regular(=fusion matrix) NIM-rep, then $\lambda \mapsto N_{\pi\lambda}$ will be equivalent to the regular NIM-rep (permute the rows and columns by π).

The exponents $\mathcal{E}(\mathcal{N}^\pi)$ of $\mathcal{N}^\pi$ is the multi-set $\pi'(\mathcal{E}(\mathcal{N}))$. If π is onto, then it can be shown using [18] that $\mathcal{N}^\pi$ will be indecomposable iff $\mathcal{N}$ is.

3.4. The diagonalising matrix U and the Pasquier algebra.

Consider now the diagonalising matrix U of (3.1c). In the event where some multiplicities m_μ are greater than 1, it will be convenient at times to introduce the following explicit notation for the entries of U: write $U_{x,(\mu,i)}$, where $1 \le i \le m_\mu$.

We would expect the diagonalising matrix U to obey essentially the same properties as S, except symmetry $S = S^t$ of course (the columns and rows are labelled by completely different sets P_+ and $\mathcal{B}$).

However, the unitary matrix U is not uniquely determined by (3.1c): for an exponent $\mu \in \mathcal{E}$ with multiplicity m_μ, we can choose for the m_μ columns corresponding to μ any orthogonal basis of the corresponding eigenspace — i.e. the freedom is parametrised for each $\mu \in \mathcal{E}$ by an $m_\mu \times m_\mu$ unitary matrix $A^{(\mu)} \in \mathrm{U}(m_\mu)$. Explicitly, an alternate matrix U' would be given by the formula

$$U'_{x,(\mu,i)} = \sum_{j=1}^{m_\mu} A_{ij}^{(\mu)} \, U_{x,(\mu,j)}$$

The question we address in this subsection is, is there a preferred choice for U which realises most of the symmetries of the S matrix which we saw in §2.1?

We claim only that the 'preferred' matrix U constructed below, diagonalises the $\mathcal{N}_\lambda$ as in (3.1c). Its relation to the change-of-coordinate matrix U, which goes from the boundary condition basis $|x\rangle$ to the Ishibashi states $|\mu\rangle\rangle$, is uncertain, although the following properties are all natural.

As mentioned in §3.3, the $\mu=0$ column can (and will) be chosen to be strictly positive. Fix any $\mu\in\mathcal{E}$. Let $\mathbb{K}_\mu$ be the number field generated by $\mathbb{Q}$ and all ratios $S_{\lambda\mu}/S_{0\mu}$, for $\lambda\in P_+$. Then for each $1\le i\le m_\mu$, we can require that all entries $U_{x,(\mu,i)}$ lie in a quadratic extension $\mathbb{K}_\mu^i$ of $\mathbb{K}$. For any Galois automorphism $\sigma\in\mathrm{Gal}(\mathbb{K}_\mu^i/\mathbb{Q})$, we can require

$$\sigma U_{x,(\mu,i)}=\epsilon_\sigma(\mu,i)\,U_{x,(\sigma\mu,i)} \tag{3.10a}$$

where $\mu\mapsto\sigma\mu$ is the permutation of (2.6), and where $\epsilon_\sigma(\mu,i)\in\{\pm1\}$. We will prove this shortly. Also, fix any vertex $1\in\mathcal{B}$; we can require U to satisfy

$$U_{x,(J\mu,i)}=e^{2\pi\mathrm{i}\,g(x)}\,U_{x,(\mu,i)}\qquad\forall J\in\mathcal{E},\ \mu\in\mathcal{E},\ x\in\mathcal{B},\ 1\le i\le m_\mu \tag{3.10b}$$

where g is the $\mathcal{N}_1$-grading associated to J by Thm.3(viii). Conversely, let $J\in P_+$ be any simple-current, and write $\mathcal{N}_{J,x}^y=\delta_{y,jx}$ for the appropriate permutation j of $\mathcal{B}$. Then each column $U_{\updownarrow,(\mu,i)}$ is an eigenvector of $\mathcal{N}_J$ with eigenvalue $e^{2\pi\mathrm{i}\,Q_J(\mu)}$, that is to say

$$U_{jx,(\mu,i)}=e^{2\pi\mathrm{i}\,Q_J(\mu)}\,U_{x,(\mu,i)} \tag{3.10c}$$

Incidentally, the relation (3.10a) allows us to prove the rationality of the coefficients of the so-called *dual Pasquier algebra* (or $\widehat{\mathcal{N}}$-algebra). Assume there is some vertex $1\in\mathcal{B}$ such that the row $U_{1,(\mu,i)}\neq0$ for all μ,i. Define [16]

$$\widehat{\mathcal{N}}_{xy}^z:=\sum_{\substack{\mu\in\mathcal{E}\\1\le i\le m_\mu}}\frac{U_{x,(\mu,i)}\,U_{y,(\mu,i)}\,U^*_{z,(\mu,i)}}{U_{1,(\mu,i)}}$$

Then for *any* such choice of $1\in\mathcal{B}$, (3.10a) tells us

$$\sigma\,\widehat{\mathcal{N}}_{xy}^z=\sum_{\substack{\mu\in\mathcal{E}\\1\le i\le m_\mu}}\frac{\epsilon_\sigma(\mu,i)\,U_{x,(\sigma\mu,i)}\,\epsilon_\sigma(\mu,i)\,U_{y,(\sigma\mu,i)}\,\epsilon_\sigma(\mu,i)\,U^*_{z,(\sigma\mu,i)}}{\epsilon_\sigma(\mu,i)\,U_{1,(\sigma\mu,i)}}=\widehat{\mathcal{N}}_{xy}^z$$

for all Galois automorphisms σ. This is precisely the statement that each coefficient $\widehat{\mathcal{N}}_{xy}^z$ is rational. This result is new, although it had been empirically observed in e.g. [16] that the coefficients $\widehat{\mathcal{N}}_{xy}^z$ for each of the then-known NIM-reps always seemed to be rational.

Ideally, we would like the coefficients $\widehat{\mathcal{N}}$ to be nonnegative integers. In this case the Perron-Frobenius eigenvalue of $\widehat{\mathcal{N}}_x$ would be given by $U_{x,(0,1)}/U_{1,(0,1)}$, and hence we would have the inequalities

$$U_{1,(0,1)}\le U_{x,(0,1)} \tag{3.11a}$$

$$|U_{x,(\mu,i)}/U_{1,(\mu,i)}|\le U_{x,(0,1)}/U_{1,(0,1)} \tag{3.11b}$$

In particular, (3.11a) is the statement that a normal $\mathbb{Z}_\ge$-matrix $A\neq0$ must have $r(A)\ge1$, and (3.11b) says that whenever $A\ge0$ then $|s|\le r(A)$ for any eigenvalue s of A. The inequality (3.11a) justifies the empirical rule of [16] for choosing the vertex $1\in\mathcal{B}$.

For example, consider the $\mathrm{sl}(2)_{16}$ NIM-rep called E_7: its diagonalising matrix U is given in [6]. We can see by inspection that its dual Pasquier coefficients cannot all be in $\mathbb{Z}_{\geq}$. In particular, (3.11a) identifies the vertex 1, and then we find (3.11b) is not satisfied.

On the other hand, the coefficients of the *Pasquier algebra* (or $\mathcal{M}$ algebra) [16]

$$\mathcal{M}_{(\lambda,i),(\mu,j)}^{(\nu,k)} := \sum_{x\in\mathcal{B}} \frac{U_{x,(\lambda,i)}\,U_{x,(\mu,j)}\,U^*_{x,(\nu,k)}}{U_{x,(0,1)}}$$

will in general *not* be rational — they will be rational iff the analogue of (3.10a) holds for rows. The coefficients $\mathcal{M}$ can be rational, only when all entries of U lie in a cyclotomic field (the proof in [23] for S works here). We will return to this shortly.

See e.g. [6] for a discussion of the (dual) Pasquier algebra. Note that our matrix U is denoted there by ψ, and our $\mathcal{B}$ is denoted there by $\mathcal{V}$. It has appeared in other related contexts — see e.g. the *classifying algebra* in e.g. [14] and references therein. Unlike fusion coefficients, neither the coefficients $\mathcal{M}$ nor $\widehat{\mathcal{N}}$ need be integral or nonnegative, and both depend on the choice of U.

To see (3.10a), first note that finding an orthogonal basis of eigenvectors for the μ-eigenspace amounts to solving a system of linear equations with coefficients in the cyclotomic field $\mathbb{K}_\mu$. Find any such basis $\vec{u}_{(\mu,i)}$, so that each 1-coordinate $(\vec{u}_{(\mu,i)})_1$ is rational. Then hit these vectors $\vec{u}_{(\mu,i)}$ componentwise by σ to yield an orthogonal basis of eigenvectors for the $\sigma(\mu)$-eigenspace. Note that $\sigma(\mu)=\mu$ iff the automorphism σ is trivial in $\mathbb{K}_\mu$, so these bases will be well-defined. When $\sigma(\mu)=J\mu$ for some simple-current $J\in\mathcal{E}$, then (3.10b) will be automatic; otherwise note from the proof of Thm.3(viii) given in the appendix that the vectors $(\vec{u}^J_{(\mu,i)})_x := e^{2\pi\mathrm{i}\,g(x)}(\vec{u}_{(\mu,i)})_x$ are orthogonal eigenvectors for $J\mu$. Run this construction through a set of representatives μ of the orbits in $\mathcal{E}$ of the group $\langle \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\mathcal{E}_{sc}\rangle$; normalising the resulting eigenbases (this is where the quadratic extensions $\mathbb{K}^i_\mu$ and the signs ϵ_σ arise), gives a unitary diagonalising matrix U satisfying (3.10).

Unlike the entries of S, those of U will *not* in general lie in a cyclotomic field, and there won't in general be a Galois action on the *rows* of U. A simple example of this is the $\mathrm{sl}(2)_{10}$ exceptional called E_6, whose diagonalising matrix is [6]

$$U=\frac{1}{2}\begin{pmatrix} a & 1 & b & b & 1 & a\\ b & 1 & a & -a & -1 & -b\\ c & 0 & -d & -d & 0 & c\\ b & -1 & a & -a & 1 & -b\\ a & -1 & b & b & -1 & a\\ d & 0 & -c & c & 0 & -d\end{pmatrix}$$

where a,b equal $\sqrt{(3\mp\sqrt{3})/6}$, respectively, and c,d equal $\sqrt{2}a,\sqrt{2}b$, respectively. Note first that $\sqrt{3+\sqrt{3}}$ does not lie in any cyclotomic field, and so neither do a,b,c,d. In fact the smallest normal extension of $\mathbb{Q}$ containing $\sqrt{3+\sqrt{3}}$ is $\mathbb{Q}[\sqrt{2},\sqrt{3+\sqrt{3}}]$ (note that $\sqrt{3+\sqrt{3}}\,\sqrt{3-\sqrt{3}}=\sqrt{3}\,\sqrt{2}$), and the corresponding Galois group is the nonabelian quaternion group $Q_8 = \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$. The Galois automorphism sending $\sqrt{2}$ to itself and $\sqrt{3 \pm \sqrt{3}}$ to $\sqrt{3 \mp \sqrt{3}}$ interchanges for instance columns 1 and 3 with $\epsilon = -1$, but doesn't send the first row anywhere. (U here is unique, up to phases for each column; no choice of phases however will give us a cyclotomic field.)

Of course, the simplest and most important example of a Galois automorphism is complex conjugation $z \mapsto z^*$. Eq.(3.10a) becomes

$$U^*_{x,(\mu,i)} = U_{x,(C\mu,i)} \tag{3.12a}$$

where $\mu \mapsto C\mu$ is charge-conjugation (2.1) — the parity $\epsilon_*(\mu, i)$ in (3.10a) will be $+1$ here because the normalisation of the columns of U only involves rescaling by a real number. Using the facts that U is unitary and C is an involution, we get that U^tU is a permutation matrix:

$$(U^tU)_{(\mu,i),(\nu,j)} = \delta_{\nu,C\mu}\, \delta_{j,i} \tag{3.12b}$$

In many examples, the analogue of (3.12a) for rows also holds: that is, there is an invertible involution ι of $\mathcal{B}$ such that

$$U^*_{x,(\mu,i)} = U_{\iota x,(\mu,i)} \tag{3.12c}$$

When this holds, we get $\mathcal{N}_{C\lambda,\iota x}^{\iota y} = \mathcal{N}_{\lambda x}^{y}$ and $(UU^t)_{xy} = \delta_{y,\iota x}$. Since $\mathrm{Tr}(U^tU) = \mathrm{Tr}(UU^t)$, the number of fixed-points of ι would equal the number of $\mu \in \mathcal{E}$ with $C\mu = \mu$, counting multiplicities. It is easy to show that ι exists iff the NIM-rep $\lambda \mapsto \mathcal{N}_{C\lambda}$ is equivalent to $\lambda \mapsto \mathcal{N}_\lambda$ — even when ι doesn't exist, they will be *linearly equivalent.* Also, ι exists iff the corresponding Pasquier algebra $\mathcal{M}$ has *real* structure constants. The existence of ι is assumed in the axioms of [7,16] and it holds in all examples of NIM-reps known to this author, but probably NIM-reps without an ι can be found for $\mathrm{sl}(3)_k$ or $\mathrm{sl}(4)_k$.

4. The current algebras at level 1

In the next two sections we obtain several new NIM-rep classifications for the current algebras, and compare them to the corresponding modular invariant classifications.

We begin in §4.1 by finding all NIM-reps for any modular data obeying the restrictive property that all primaries are simple-currents. This allows us immediately to do all simply-laced current algebras at level 1. The NIM-reps for the $B^{(1)}$- and $C^{(1)}$-series at level 1 follow from the $\widehat{\mathrm{sl}}(2)$ classification, so we repeat the $\widehat{\mathrm{sl}}(2)$ classification in §4.3.

In all these cases, the NIM-rep and modular invariant classifications match up fairly well: each modular invariant has a unique NIM-rep, and most NIM-reps are paired with a unique modular invariant. The only interesting situation here is $\mathrm{so}(8n)_1$, where different modular invariants correspond to identical NIM-reps.

Note that NIM-reps (unlike modular invariants) depend only on the fusion ring. When two fusion rings are isomorphic, their NIM-reps will be identical. In [29] we found all isomorphisms $X_{r,k} \cong X'_{r',\ell'}$ among the fusion rings of current algebras. The complete list is: $\mathrm{sp}(2n)_k \cong \mathrm{sp}(2k)_n$ for all n, k; all $\mathrm{so}(2n+1)$ at level 1 are isomorphic to $\mathrm{sl}(2)_2 \cong \mathrm{sp}(4)_1 \cong E_{8,2}$; $\mathrm{sl}(2)_k \cong \mathrm{sp}(2k)_1$; $\mathrm{so}(2n)_1 \cong \mathrm{so}(2m)_1$ whenever $n \equiv m \pmod 2$, and in addition odd

m are isomorphic to $\mathrm{sl}(2)_2$; $\mathrm{sl}(3)_1 \cong E_{6,1}$; $\mathrm{sl}(2)_1 \cong E_{7,1}$; $F_{4,1} \cong G_{2,1}$; $F_{4,3} \cong G_{2,4}$; and finally $E_{8,3} \cong F_{4,2}$.

Coincidentally, when the fusion rings of $X_{r,k}$ and $X'_{r',k'}$ are isomorphic, it turns out that their modular invariant classifications will usually be identical. The only exception is $\mathrm{so}(4n)_1$, which has either 2 or 6 modular invariants, depending on whether or not n is odd.

4.1. All primaries are simple-currents.

The simple-currents (i.e. the primaries with quantum-dimension 1 — see §2.1) always form an abelian group, called the *centre* of the modular data. Any NIM-rep, when restricted to the centre, yields a group-representation of the centre by permutation matrices. In this subsection we consider the special case where all primaries $\lambda \in P_+$ are simple-currents (the modular data though is otherwise general — it may or may not come from a current algebra).

Proposition 4. Consider any modular data. Suppose all primaries in P_+ are simple-currents.

(a) The indecomposable NIM-reps are in one-to-one correspondence with the subgroups $\mathcal{J}$ of the centre: $\mathcal{J} \leftrightarrow \mathcal{N}(\mathcal{J})$. The exponent $\mathcal{E}$ of the NIM-rep $\mathcal{N}(\mathcal{J})$ is $\mathcal{J}$. (We will explicitly construct $\mathcal{N}(\mathcal{J})$ below.) The NIM-rep is uniquely specified by its exponents.

(b) The exponent of any modular invariant is a subgroup of the centre. Thus any modular invariant is NIMmed. However, some subgroups (hence NIM-reps) may be realised by none or by several modular invariants. There may be more/less/the same number of modular invariants as NIM-reps.

In particular, choose any subgroup $\mathcal{J}$ of the centre P_+, and put $k = \|\mathcal{J}\|$. Define a k-dimensional NIM-rep as follows. Let $\mathcal{J}'$ be the subset (in fact subgroup) of P_+, consisting of all primaries J' for which $Q_J(J') \in \mathbb{Z}$ for all $J \in \mathcal{J}$. There will be $\|P_+\|/k$ such J'. This is a subgroup because of the relation $Q_J(J'J'') = Q_J(J') + Q_J(J'')$ which holds for any simple-currents J, J', J'', and which follows immediately from (2.4). Now consider the quotient group $P_+/\mathcal{J}' = \{[J_0], [J_1], \ldots, [J_{k-1}]\}$. It will in fact be isomorphic to $\mathcal{J}$. Define the NIM-rep $\mathcal{N}(\mathcal{J})$ by

$$(\mathcal{N}(\mathcal{J})_J)_{ij} = \delta_{[JJ_i],[J_j]} \qquad \forall J \in P_+$$

So the rows and columns of $\mathcal{N}(\mathcal{J})$ are essentially labelled by the elements of $P_+/\mathcal{J}'$. To get that the exponents of $\mathcal{N}(\mathcal{J})$ are $\mathcal{J}$, use the fact that $J' \in P_+$ is sent to I iff $J' \in \mathcal{J}'$, and so $Q_J(J') \in \mathbb{Z}$ for any exponent J and any $J' \in \mathcal{J}'$.

The two extremes are when the subgroup is all of P_+, in which case the NIM-rep is given by fusion matrices, and when the subgroup is $\{0\}$, in which case the NIM-rep is the constant $\mathcal{N}_J = 1$.

It is clear from Thm.1(iii) and Thm.3(iii) that the exponents of a modular invariant and a NIM-rep must both form a subgroup of the centre P_+. It is not obvious that there is only one NIM-rep realising that subgroup. To see the general argument, it is perhaps easiest to consider an example: $P_+ \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \cong \mathcal{J}$. Let J_1, J_2, J_3 be the corresponding generators. Let $\mathcal{N}$ be any NIM-rep with exponents P_+. We know from Thm.3(x) that

$\mathrm{Tr}(\mathcal{N}_J) = 0$ provided $J \neq 0$, so the permutation associated to $\mathcal{N}_J$, for any $J \neq 0$, can have no fixed-points. Thus the permutation associated to $\mathcal{N}_{J_1}$ must be a disjoint product of nine 4-cycles. By relabelling the rows/columns appropriately, we may take it to send $i + 4j + 12k$ $(i \in \mathbb{Z}_4, j \in \mathbb{Z}_3, k \in \mathbb{Z}_3)$ to $(i + 1 \,(\mathrm{mod}\, 4)) + 4j + 12k$. Likewise, $\mathcal{N}_{J_2}$ must be a disjoint product of 12 3-cycles, and it must commute with $\mathcal{N}_{J_1}$, so we may take the corresponding permutation to send $i+4j+12k$ to $i+4(j+1\,(\mathrm{mod}\,3))+12k$. The matrix $\mathcal{N}_{J_3}$ is handled similarly; none of its 3-cycles can coincide with those of $\mathcal{N}_{J_2}$ because otherwise $\mathcal{N}_{J_3}\mathcal{N}_{J_2}^{-1} = \mathcal{N}_{J_3 J_2^{-1}}$ would have fixed points and nonzero trace. So we can likewise fix $\mathcal{N}_{J_3}$. Manifestly, the resulting NIM-rep is the regular NIM-rep corresponding to the fusion matrices.

4.2. The simply-laced algebras at level 1.

The algebra $\widehat{\mathrm{sl}}(n) = A^{(1)}_{n-1}$, $n \geq 2$, at level 1 has n primaries, $P_+ = \{0, \Lambda_1, \ldots, \Lambda_{n-1}\}$. Put $\Lambda_0 = 0$, then $\Lambda_i = J^i$ for the simple-current $J = \Lambda_1$. The centre of $\mathrm{sl}(n)_1$ is the cyclic group $\mathbb{Z}_n$, so there is an indecomposable NIM-rep corresponding to each divisor d of n. In particular, the exponents will be generated by J^d, the subgroup $\mathcal{J}'$ defined above will be generated by $J^{n/d}$, and the resulting NIM-rep will be n/d-dimensional. This classification is given in [6].

There is a modular invariant, namely $M[J^d]$ in (2.9), for any divisor d of n for which $(n-1)d$ is even [36]. It has exponents $\langle J^{n/d}\rangle$ and corresponds to the NIM-rep $\mathcal{N}(\langle J^{n/d}\rangle)$.

The algebra $\widehat{\mathrm{so}}(2r) = D^{(1)}_r$, $r \geq 4$, at level 1 has 4 primaries $P_+ = \{0, J_v = \Lambda_1, J_s = \Lambda_r, J_c = \Lambda_{r-1}\}$, all of which are simple-currents. For r odd they define the cyclic group $\langle J_s\rangle \cong \mathbb{Z}_4$, while for r even they define the group $\langle J_v, J_s\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus there are precisely three indecomposable NIM-reps for r odd — one for each choice of exponents $\mathcal{E} = \{0\}, \{0, J_v\}, \{0, J_v, J_s, J_c\}$. For r even, there are precisely five indecomposable NIM-reps — one for each choice of exponents

$$\mathcal{E} = \{0\},\ \{0, J_v\},\ \{0, J_s\},\ \{0, J_c\},\ \{0, J_v, J_s, J_c\}$$

For $D_{r,1}$, when 4 does not divide r, there are only two modular invariants [26]: $M = I$ (which has exponents $\{0, J_v, J_s, J_c\}$) and $M = C_1$, the permutation fixing 0 and Λ_1 and interchanging $\Lambda_r \leftrightarrow \Lambda_{r-1}$ (which has exponents $\{0, J_v\}$). When 4 divides r, there are six modular invariants [26]: along with I and C_1, these are $M[J_s]$, $C_1\,M[J_s]$, $M[J_s]\,C_1$, and $C_1\,M[J_s]\,C_1$ (with exponents $\{0, J_s\}, \{0\}, \{0\}, \{0, J_c\}$, resp.). In particular, both

$$C_1\,M[J_s] = \left(\chi_0 + \chi_{\Lambda_{r-1}}\right)\left(\chi_0^* + \chi_{\Lambda_r}^*\right)\,, \qquad M[J_s]\,C_1 = \left(\chi_0 + \chi_{\Lambda_r}\right)\left(\chi_0^* + \chi_{\Lambda_{r-1}}^*\right)$$

correspond to the identical NIM-rep (namely $\mathcal{N}_J = 1\ \forall J$).

The algebra $E_{6,1}$ has centre $\{0, \Lambda_1, \Lambda_5\} \cong \mathbb{Z}_3$, two indecomposable NIM-reps, and two modular invariants ($M = I$ and $M = C$). The algebra $E_{7,1}$ has centre $\{0, \Lambda_6\} \cong \mathbb{Z}_2$, two indecomposable NIM-reps, and one modular invariant ($M = I$). The algebra $E_{8,1}$ has trivial centre $\{0\}$, one indecomposable NIM-rep, and one modular invariant.

4.3. The algebra $\widehat{\mathrm{sl}}(2) = A_1^{(1)}$, *at level* k.

Because we'll be needing it in the next two subsections, we repeat here the NIM-rep classification for $\widehat{\mathrm{sl}}(2)$, which was first given in [7].

Let $\mathcal{N}$ be any indecomposable NIM-rep of $A_{1,k}$. Its modular data is given in §2.1. A fusion generator for $A_{1,k}$ is Λ_1, so it suffices to give $\mathcal{N}_1 = \mathcal{N}_{\Lambda_1}$. For k odd, the fusion graph for $\mathcal{N}_1$ is either A_{k+1} or the tadpole $T_{(k+1)/2}$ (see Figure 1). For k even, the possible fusion graphs are A_{k+1} and $D_{k/2+2}$, except for $k = 10, 16$ or 28 where in addition there are E_6, E_7, E_8 respectively.

The modular invariants for $A_{1,k}$ were found in [27]. Each corresponds to a unique NIM-rep, namely one of A-D-E type, as is well-known.

4.4. The algebra $\widehat{\mathrm{so}}(2r+1) = B_r^{(1)}$, *for* $r \geq 3$ *at level* 1.

The weights here are $P_+ = \{0, \Lambda_1, \Lambda_r\}$. For $B_{r,1}$ the only modular invariant [26] is the identity matrix I. We learned above that its fusion ring is isomorphic to that of $\mathrm{sl}(2)_2$ (the isomorphism sends Λ_r to the fusion generator Λ_1 of $\mathrm{sl}(2)_2$) and so we can read off its NIM-reps from the classification of §4.3: we find that there is only the 'regular' one, given by the fusion matrices, which assigns to the generator Λ_r the fusion graph A_3.

4.5. The algebra $\widehat{\mathrm{sp}}(2r) = C_r^{(1)}$, *for* $r \geq 2$ *at level* 1.

Here, $P_+ = \{0, \Lambda_1, \ldots, \Lambda_r\}$. Write Λ_0 for 0. The fusion-isomorphism between $C_{r,1}$ and $A_{1,r}$ identifies the primary Λ_i of $C_{r,1}$ with the primary $i\Lambda_1$ of $A_{1,r}$. The NIM-reps for $C_{r,1}$ are thus of A-D-E or tadpole type, exactly as in §4.3.

The modular invariants for $C_{r,1}$ [26] fall into the A-D-E pattern, and are in a natural one-to-one correspondence with those of $A_{1,r}$ (again using the identification $\Lambda_i \leftrightarrow i\Lambda_1$).

Thus the NIM-rep $\leftrightarrow$ modular invariant situation for $C_{r,1}$ is identical to that of $A_{1,r}$.

4.6. The algebras $G_2^{(1)}$ *and* $F_4^{(1)}$ *at level* 1.

$G_{2,1}$ has $P_+ = \{0, \Lambda_2\}$. We compute the quantum-dimension: $\frac{S_{\Lambda_2,0}}{S_{0,0}} = \frac{1+\sqrt{5}}{2}$, the Golden Mean. Thus there's a Galois automorphism σ for which $\sigma\frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} = \frac{-2}{1+\sqrt{5}}$. Applying that σ to the quantum-dimension and using (2.6), we see that $\sigma 0 = \Lambda_2$. Thus for any (indecomposable) NIM-rep of $G_{2,1}$, $m_{\Lambda_2} = m_0 = 1$, and the NIM-rep must be 2-dimensional. It is now trivial to find it:

$$\mathcal{N}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{N}_{\Lambda_2} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and the fusion graph of Λ_2 is the tadpole T_2.

The only modular invariant [26] is $M = I$, which is paired with T_2.

The situation is completely identical for $F_{4,1}$: $P_+ = \{0, \Lambda_4\}$ here, and the fusion-isomorphism identifies Λ_4 with Λ_2. There is again only one NIM-rep and one modular invariant, and again the graph is the tadpole T_2.

5. The unitary and orthogonal algebras at level 2

5.1. $\widehat{\mathrm{sl}}(n)$ *at level* 2.

Consider next $\widehat{\mathrm{sl}}(n)=A_{n-1}^{(1)}$ at level 2. The weights λ are all of the form $\lambda(ab):=\Lambda_a+\Lambda_b$, for $0\le a,b<n$. Since $\lambda(ab)=\lambda(ba)$, we will usually require $a\le b$.

The simple-current J and charge-conjugation C act on P_+ by:

$$J\lambda(ab)=\lambda(a+1,b+1)\ ,\qquad C\lambda(ab)=\lambda(n-b,n-a)$$

J has order n. For any divisor d of n, we get the modular invariant $M[J^d]$ for $\mathrm{sl}(n)_2$ given in (2.9), where $Q_{J^d}(\lambda(ab))=d\,(a+b)/n$ and $R_{J^d}=2d$. For example, $M[J^n]=I$ and $M[J]=C$.

The remaining, exceptional, $\mathrm{sl}(n)_2$ modular invariants $\mathcal{E}^{(n,2)}$ are [37]

$$\begin{aligned}
\mathcal{E}^{(10,2)}&=\sum_{i=0}^{9}|\chi_{\lambda(i,i)}+\chi_{\lambda(i+3,i+7)}|^2+\sum_{i=0}^{4}|\chi_{\lambda(i,i+3)}+\chi_{\lambda(i+5,i+8)}|^2\\
\mathcal{E}^{(16,2)}&=\sum_{i=0}^{7}\big(|\chi_{\lambda(i,i)}+\chi_{\lambda(i+8,i+8)}|^2+|\chi_{\lambda(i,i+4)}+\chi_{\lambda(i+8,i-4)}|^2+|\chi_{\lambda(i,i+8)}|^2\\
&\qquad+|\chi_{\lambda(i,i+6)}+\chi_{\lambda(i+8,i-2)}|^2+\big(\chi_{\lambda(i+3,i+5)}+\chi_{\lambda(i-5,i-3)}\big)\,\chi^*_{\lambda(i,i+8)}\\
&\qquad+\chi_{\lambda(i,i+8)}\,(\chi_{\lambda(i+3,i+5)}+\chi_{\lambda(i-5,i-3)})^*\,\big)\\
\mathcal{E}^{(28,2)}&=\sum_{i=0}^{13}\big(|\chi_{\lambda(i,i)}+\chi_{\lambda(i+14,i+14)}+\chi_{\lambda(i+5,i-5)}+\chi_{\lambda(i-9,i+9)}|^2\\
&\qquad+|\chi_{\lambda(i+3,i-3)}+\chi_{\lambda(i-11,i+11)}+\chi_{\lambda(i+6,i-6)}+\chi_{\lambda(i-8,i+8)}|^2\big)
\end{aligned}$$

together with the matrix products $C\cdot\mathcal{E}^{(10,2)}$, $C\cdot\mathcal{E}^{(16,2)}$, $\frac{1}{2}M[J^4]\cdot\mathcal{E}^{(16,2)}$, and $C\cdot\mathcal{E}^{(28,2)}$.

Note the strong resemblance of the exceptional modular invariants here to the so-called E_6,E_7,E_8 exceptionals of $\widehat{\mathrm{sl}}(2)$ [27]. This is not a coincidence, and is a consequence of a duality between $\widehat{\mathrm{sl}}(n)$ level k, and $\widehat{\mathrm{sl}}(k)$ level n. See also the resemblance between (A.1) and the S matrix for $\widehat{\mathrm{sl}}(2)$ level n.

We next turn to the NIM-reps. The proof that our list is complete, is given in §A.1. Write $n=2^hm$ where m is odd. We know $J=\lambda(11)$ and $\Lambda_1=\lambda(01)$ are fusion-generators, so so are J^m, J^{2^h} and $\lambda:=J^{(m-1)/2}\Lambda_1=\lambda(\frac{m-1}{2},\frac{m+1}{2})$. Thus, the NIM-rep is uniquely defined once the matrices $A:=\mathcal{N}_\lambda$, $P':=\mathcal{N}_{J^m}$ and $P'':=\mathcal{N}_{J^{2^h}}$ are known. The reason it is more convenient to use these fusion-generators is Lemma A in the appendix — roughly, the matrix A is nearly symmetric, and its failure to be symmetric is governed by the permutation matrix P'.

The matrix A will correspond to the disjoint union of equivalent diagrams taken from Figure 3. Each of those diagrams corresponds to a digraph, as follows. Number the diagram nodes from 1 to n, say from left to right, top to bottom. The weight (k or $2k$) of each node tells how many vertices are represented by that node. So each vertex in the